Chapter Outline

2.1 Arithmetic Sequences
2.2 Solution Sets and Graphs of Linear Equations
2.3 Determining the Equation of a Line
2.4 Forms of Linear Equations
2.5 Linear Function as a Model
2.6 Graphing Linear Inequalities in Two Variables
2.7 Solving Linear Systems by Graphing
2.8 Solving Systems Using Substitution
2.9 Solving Linear Systems by Elimination
2.10 Comparing Methods for Solving Linear Systems
2.11 Applications of Linear Systems
2.12 Linear Programming
2.13 Graphs of Absolute Value Equations
2.14 Body Temperature (Absolute Value Inequalities)
Here you will identify an arithmetic sequence and its common difference. You will also write recursive and explicit formulas for a sequence given the common difference and a term.

Halley’s Comet appears in the sky approximately every 76 years. The comet was first spotted in the year 1531. Find the $n^{th}$-term rule and the $10^{th}$-term for the sequence represented by this situation.

**Review**

Find the pattern: 3, 5, 7, 9, ....

What is the 5th term of the sequence?

What is the 20th term of the sequence?

**What is an Arithmetic Sequence?**

In this concept we will begin looking at a specific type of sequence called an arithmetic sequence. In an arithmetic sequence the difference between any two consecutive terms is constant. This constant difference is called the common difference. For example, question one in the Review Queue above is an arithmetic sequence. The difference between the first and second terms is $(5 - 3) = 2$, the difference between the second and third terms is $(7 - 5) = 2$ and so on.

**Finding Formulas for Arithmetic Sequences**

Before proceeding, we need some notation:

We use $n$ to represent the "term number" for a particular number in the sequence. For example, in the Review sequence, $n = 1$ refers to the first term which is 3. Similarly, $n = 2$ refers to the second term which is 5.

We use $a_n$ to represent the $n$th term of the sequence. For example, $a_2$ represents the second term of the sequence so $a_2 = 3$. Similarly, $a_4 = 9$ since 9 is the fourth term of the sequence.

Since $a_n$ represents the $n$th term, $a_{n-1}$ would represent the previous term. Think about it. You will see this notation used quite a bit below.

**The RECURSIVE FORMULA**

The word recursive refers to a process that repeats the same steps over and over again. An arithmetic sequence can be thought of as a recursive process since you are repeatedly adding the same number over and over again.

A recursive formula for a sequence tells you what you must do to the current term to get to the next term. In an arithmetic sequence, this means you must ADD the constant difference. In the review example, no matter which
term you start from, you add 2 to get to the next term. If you wanted to find the 5th term, you would start from the 4th term. Then simply add 2. This would look like:

\[ a_5 = a_4 + 2 \]
\[ a_5 = 9 + 2 \]
\[ a_5 = 11 \]

We can generalize this process for this sequence replacing \( a_5 \) with \( a_n \) (the term we wish to calculate) and \( a_4 \) with \( a_{n-1} \) (the previous term).

This gives us the formula:

\[ a_n = a_{n-1} + 2 \]

But we are missing something. This formula alone is not enough. We need a starting point – we need to define the first term of the sequence. So, the full recursive formula for this sequence is:

\[ a_1 = 3 \]
\[ a_n = a_{n-1} + 2 \]

It is EXTREMELY important that you define the first term AND write the recursive formula for the \( a_n \) term.

---

**The Recursive Formula for an Arithmetic Sequence**

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>( a_1 = \text{first term} )</td>
</tr>
<tr>
<td></td>
<td>( a_n = a_{n-1} + d )</td>
</tr>
</tbody>
</table>

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**The EXPLICIT FORMULA**

There is one BIG flaw with the recursive formula. Can you figure out what is the flaw? Say you wanted to know the 100th term of the review sequence? In order to do so using the recursive formula, which terms would you need to know?

An explicit formula does not rely on the previous term. An explicit formula will give you ANY term of the sequence simply by substituting in the term number.

Recall that we can find the common difference for a sequence by subtracting any two terms in a row. We can write this as:

\[ a_n - a_{n-1} = d \], where \( a_{n-1} \) and \( a_n \) represent two consecutive terms and \( d \) represents the common difference.
Since the same value, the common difference, \( d \), is added to get each successive term in an arithmetic sequence we can determine the value of any term from the first term and how many times we need to add \( d \) to get to the desired term as illustrated below:

Given the sequence: 3, 5, 7, 9, \ldots\) in which \( a_1 = 3 \) and \( d = 2 \)

\[
\begin{align*}
a_1 &= 3 + (0)(2) = 3 + 0 = 3 \\
a_2 &= 3 + (1)(2) = 3 + 2 = 5 \\
a_3 &= 3 + (2)(2) = 3 + 4 = 7 \\
a_4 &= 3 + (3)(2) = 3 + 6 = 9 \\
\vdots \\
a_n &= 3 + (n - 1)(2) \\
a_n &= 3 + 2n - 2 \\
a_n &= 2n + 1
\end{align*}
\]

**The Explicit Formula for an Arithmetic Sequence**

\[
a_n = a_1 + (n - 1)(d)
\]

**Example A**

Find the common difference and a recursive and explicit formula for the arithmetic sequence: 2, 5, 8, 11\ldots \)

**Solution:** To find the common difference we subtract consecutive terms.

\[
\begin{align*}
5 - 2 &= 3 \\
8 - 5 &= 3, \text{ thus the common difference is } 3. \\
11 - 8 &= 3
\end{align*}
\]

The recursive formula for this sequence is:

\[
\begin{align*}
a_1 &= 2 \\
a_n &= a_{n-1} + 3
\end{align*}
\]

The explicit formula for this sequence is:

\[
\begin{align*}
a_n &= 2 + (n - 1)(3) \\
 &= 2 + 3n - 3 \\
 &= 3n - 1 \\
So, \; a_n &= 3n - 1.
\end{align*}
\]
Example B

Find the explicit formula and the 100th term for the arithmetic sequence in which \(a_1 = -9\) and \(d = 2\).

**Solution:** We have what we need to plug into the rule:

\[
\begin{align*}
a_n &= -9 + (n - 1)(2) \\
&= -9 + 2n - 2 \\
&= 2n - 11,
\end{align*}
\]

Thus, the \(n^{th}\) term rule is \(a_n = 2n - 11\).

Now to find the 100th term we can use our rule and replace \(n\) with 100:

\[
a_{100} = 2(100) - 11 = 200 - 11 = 189.
\]

Example C

Find the recursive formula, the explicit, and the 100th term for the arithmetic sequence in which \(a_3 = 8\) and \(d = 7\).

**Solution:** This one is a little less straightforward as we will have to first determine the first term from the term we are given. To do this, we will replace \(a_n\) with \(a_3 = 8\) and use 3 for \(n\) in the explicit formula to determine the unknown first term as shown:

\[
\begin{align*}
a_1 + (3 - 1)(7) &= 8 \\
a_1 + 2(7) &= 8 \\
a_1 + 14 &= 8 \\
a_1 &= -6
\end{align*}
\]

Now that we have the first term and the common difference we can follow the same process used in the previous example to find the explicit formula:

\[
\begin{align*}
a_n &= -6 + (n - 1)(7) \\
&= -6 + 7n - 7 \\
&= 7n - 13
\end{align*}
\]

Now we can find the 100th term: \(a_{100} = 7(100) - 13 = 687\).

Finally, we can write the recursive formula:

\[
\begin{align*}
a_1 &= -6 \\
a_n &= a_{n-1} + 7.
\end{align*}
\]

Intro Problem Revisit From the information given, we can conclude that \(a_1 = 1531\) and \(d = 76\).

We now have what we need to plug into the rule:

\[
\begin{align*}
a_n &= 1531 + (n - 1)(76) \\
&= 1531 + 76n - 76 \\
&= 76n + 1455
\end{align*}
\]
Now to find the $10^{th}$ term we can use our rule and replace $n$ with 10: $a_{10} = 76(10) + 1455 = 760 + 1455 = 2215$.

**Guided Practice**

1. Find the common difference and the $n^{th}$ term rule for the sequence: $5, -3, -11, \ldots$
2. Write the explicit formula and find the $45^{th}$ term for the arithmetic sequence with $a_{10} = 1$ and $d = -6$.
3. Find the $62^{nd}$ term for the arithmetic sequence with $a_1 = -7$ and $d = \frac{3}{2}$.

**Answers**

1. The common difference is $-3 - 5 = -8$. Now $a_n = 5 + (n - 1)(-8) = 5 - 8n + 8 = -8n + 13$.
2. To find the first term:

   $$a_1 + (10 - 1)(-6) = 1$$
   $$a_1 - 54 = 1$$
   $$a_1 = 55$$

   Find the $n^{th}$ term rule: $a_n = 55 + (n - 1)(-6) = 55 - 6n + 6 = -6n + 61$.

   Finally, the $45^{th}$ term: $a_{45} = -6(45) + 61 = -209$.

3. This time we will not simplify the $n^{th}$ term rule, we will just use the formula to find the $62^{nd}$ term: $a_{62} = -7 + (62 - 1)\left(\frac{3}{2}\right) = -7 + 61\left(\frac{3}{2}\right) = -\frac{14}{2} + \frac{183}{2} = \frac{169}{2}$.

**Vocabulary**

**Arithmetic Sequence**

A sequence in which the difference between any two consecutive terms is constant.

**Common Difference**

The value of the constant difference between any two consecutive terms in an arithmetic sequence.

**Recursive Formula**

A formula that finds a term of a sequence using the previous term.

**Explicit Formula**

A formula that finds the $n^{th}$ term of a sequence using the first term, the term number, and the common difference.

**Practice**

Identify which of the following sequences is arithmetic. If the sequence is arithmetic find the recursive AND explicit formulae for the sequence.

1. 2, 3, 4, 5, \ldots
2. 6, 2, -1, -3, \ldots
3. 5, 0, -5, -10, \ldots
4. 1, 2, 4, 8, \ldots
5. 0, 3, 6, 9, \ldots
2.1. Arithmetic Sequences

6. 13, 12, 11, 10, …
7. 4, −3, 2, −1, …
8. \(a, a+2, a+4, a+6, \ldots\)

For 9-15. Write a recursive and an explicit formula for each arithmetic sequence with the given term and common difference.

9. \(a_1 = 15\) and \(d = -8\)
10. \(a_1 = -10\) and \(d = \frac{1}{2}\)
11. \(a_3 = 24\) and \(d = -2\)
12. \(a_5 = -3\) and \(d = 3\)
13. \(a_{10} = -15\) and \(d = -11\)
14. \(a_7 = 32\) and \(d = 7\)
15. \(a_{n-2} = 3n + 2,\) find \(a_n\)

16. Suppose an arithmetic sequence has a recursive formula of:

\[
\begin{align*}
a_1 &= 15 \\
av_n &= a_{n-1} - 6
\end{align*}
\]

(a) Find an explicit formula for this sequence
(b) Find the value of \(a_{60}\)

Suppose an arithmetic sequence has a recursive formula of:

\[
\begin{align*}
a_1 &= -2 \\
av_n &= a_{n-1} + 10
\end{align*}
\]

17. (a) Find an explicit formula for this sequence.
    (b) Find \(a_{40}\) for this sequence.

18. Suppose an arithmetic sequence has an explicit formula of \(a_n = 7n - 5.\)

(a) Find a recursive formula for this sequence.
(b) Find the value of \(a_{100}\).

19. Suppose an arithmetic sequence has an explicit formula of \(a_n = -4n + 5.\)

(a) Find a recursive formula for this sequence.
(b) Find the value of \(a_{80}\).

20. Create a graph of the sequences indicated below. Your x-axis should be \(n\) (the term number), and your y-axis should \(a_n\) (the actual term).

(a) The sequence given in number 1 above.
(b) The sequence given in number 2 above.
(c) The sequence given in number 3 above.
(d) The sequence given in number 4 above.
(e) The sequence given in number 5 above.
(f) If a sequence is arithmetic, then what shape is its graph?
2.2 Solution Sets and Graphs of Linear Equations

Here you’ll learn how to make a table of values and graph a function given the function’s rule.

In the previous section, you were asked to identify arithmetic sequences and find formulas for those sequences. In the Practice section of 2.1, you were asked to graph some arithmetic sequences. Those sequences looked like a line. In the rest of this chapter you will expand your knowledge of linear functions.

What if you were given a function rule like $y = 2x - 3$? How could you graph that function in the coordinate plane? What does the graph of the function represent? What does it mean to be a solution to the equation? After completing this Concept, you’ll be able to graph functions like this one by either creating a table of values or using its slope and intercept.

Watch This

Guidance

Once we know how to plot points (x,y) on a coordinate plane, we can think about how we’d go about plotting a relationship between x— and y—values. In the past, you have plotted points on a coordinate plane that you were given by your teacher. A set like that is a relation, and there isn’t necessarily a relationship between the x—values and y—values. If there is a relationship between the x— and y—values, and each x—value corresponds to exactly one y—value, then the relation is called a function. Remember that a function is a particular way to relate one quantity to another.

Example A

If you’re reading a book and can read twenty pages an hour, there is a relationship between how many hours you read and how many pages you read. You may even know that you could write the formula as either $n = 20h$ or $h = \frac{n}{20}$, where $h$ is the number of hours you spend reading and $n$ is the number of pages you read. To find out, for example, how many pages you could read in $3\frac{1}{2}$ hours, or how many hours it would take you to read 46 pages, you could use one of those formulas. Or, you could make a graph of the function:
Once you know how to graph a function like this, you can simply read the relationship between the $x$—and $y$—values off the graph. You can see in this case that you could read 70 pages in $3\frac{1}{2}$ hours, and it would take you about $2\frac{1}{3}$ hours to read 46 pages.

Generally, the graph of a function appears as a line or curve that goes through all points that have the relationship that the function describes. If the domain of the function (the set of $x$—values we can plug into the function) is all real numbers, then we call it a continuous function. If the domain of the function is a particular set of values (such as whole numbers only), then it is called a discrete function. The graph will be a series of dots, but they will still often fall along a line or curve.

In graphing equations, we assume the domain (possible $x$-values) is all real numbers, unless otherwise stated. Often, though, when we look at data in a table, the domain will be whole numbers (number of presents, number of days, etc.) and the function will be discrete. But sometimes we’ll still draw the graph as a continuous line to make it easier to interpret. Be aware of the difference between discrete and continuous functions as you work through the examples.

**Example B**

Sarah is thinking of the number of presents she receives as a function of the number of friends who come to her birthday party. She knows she will get a present from her parents, one from her grandparents and one each from her uncle and aunt. She wants to invite up to ten of her friends, who will each bring one present. She makes a table of how many presents she will get if one, two, three, four or five friends come to the party. Plot the points on a coordinate plane and graph the function that links the number of presents with the number of friends. Use your graph to determine how many presents she would get if eight friends show up.

<table>
<thead>
<tr>
<th>Number of Friends</th>
<th>Number of Presents</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
</tr>
</tbody>
</table>
The first thing we need to do is decide how our graph should appear. We need to decide what the independent variable is, and what the dependent variable is. Clearly in this case, the number of friends can vary independently, but the number of presents must depend on the number of friends who show up.

So we’ll plot friends on the $x$–axis and presents on the $y$–axis. Let’s add another column to our table containing the coordinates that each (friends, presents) ordered pair gives us.

<table>
<thead>
<tr>
<th>Friends ($x$)</th>
<th>Presents ($y$)</th>
<th>Coordinates $(x, y)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>4</td>
<td>(0, 4)</td>
</tr>
<tr>
<td>1</td>
<td>5</td>
<td>(1, 5)</td>
</tr>
<tr>
<td>2</td>
<td>6</td>
<td>(2, 6)</td>
</tr>
<tr>
<td>3</td>
<td>7</td>
<td>(3, 7)</td>
</tr>
<tr>
<td>4</td>
<td>8</td>
<td>(4, 8)</td>
</tr>
<tr>
<td>5</td>
<td>9</td>
<td>(5, 9)</td>
</tr>
</tbody>
</table>

Next we need to set up our axes. It is clear that the number of friends and number of presents both must be positive, so we only need to show points in Quadrant I. Now we need to choose a suitable scale for the $x$– and $y$–axes. We only need to consider eight friends (look again at the question to confirm this), but it always pays to allow a little extra room on your graph. We also need the $y$–scale to accommodate the presents for eight people. We can see that this is still going to be under 20!

The scale of this graph has room for up to 12 friends and 15 presents. This will be fine, but there are many other scales that would be equally good!

Now we proceed to plot the points. The first five points are the coordinates from our table. You can see they all lie on a straight line, so the function that describes the relationship between $x$ and $y$ will be linear. To graph the function, we simply draw a line that goes through all five points. This line represents the function.

This is a discrete problem since Sarah can only invite a positive whole number of friends. For instance, it would be impossible for 2.4 or -3 friends to show up. So although the line helps us see where the other values of the function are, the only points on the line that actually are values of the function are the ones with positive whole-number coordinates.

The graph easily lets us find other values for the function. For example, the question asks how many presents Sarah
would get if eight friends come to her party. Don’t forget that \( x \) represents the number of friends and \( y \) represents the number of presents. If we look at the graph where \( x = 8 \), we can see that the function has a \( y \)-value of 12.

**Solution**

If 8 friends show up, Sarah will receive a total of 12 presents.

**Solution Sets to Equations and Graphing a Function Given a Rule**

You may remember that the solution to a one-variable equation such as \( 2x + 3 = 5 \) is the value of \( x \) that makes the equation true. In this case, the solution is \( x = 1 \). In a two-variable equation such as \( y = -3x + 1 \) there are TWO variables. A **solution** to a two variable equation is a coordinate pair that makes the equation true. In the case of \( y = -3x + 1 \), one solution is the point \((0, 1)\). Another solution is the point \((2, -4)\). We could list many more solutions to this equation. The set of all solutions to the equation is called the **solution set**.

Of course, it would be impossible to write out ALL the solutions to an equation. Another way to represent the solution set is with a graph. So, how do we graph the solution set to an equation? The example below shows two methods for graphing a linear equation.

**Example C**

Ali is trying to work out a trick that his friend showed him. His friend started by asking him to think of a number, then double it, then add five to the result. Ali has written down a rule to describe the first part of the trick. He is using the letter \( x \) to stand for the number he thought of and the letter \( y \) to represent the final result of applying the rule. He wrote his rule in the form of an equation: \( y = 2x + 5 \).

Help him visualize what is going on by graphing the solution set to the function using the equation given.

**Method One - Construct a Table of Values**

If we wish to plot a few points to see what is going on with this function, then the best way is to construct a table and populate it with a few \((x,y)\) pairs. We’ll use 0, 1, 2 and 3 for \( x \)-values.

<table>
<thead>
<tr>
<th>( x )</th>
<th>( y )</th>
</tr>
</thead>
<tbody>
<tr>
<td>0</td>
<td>5</td>
</tr>
<tr>
<td>1</td>
<td>7</td>
</tr>
<tr>
<td>2</td>
<td>9</td>
</tr>
<tr>
<td>3</td>
<td>11</td>
</tr>
</tbody>
</table>

Next, we plot the points and join them with a line.
This method is nice and simple—especially with linear relationships, where we don’t need to plot more than two or three points to see the shape of the graph. In this case, the function is continuous because the domain is all real numbers—that is, Ali could think of any real number, even though he may only be thinking of positive whole numbers.

**Method Two - Intercept and Slope**

Another way to graph this function (one that we’ll learn in more detail in a later lesson) is the slope-intercept method. To use this method, follow these steps:

1. Find the $y$ value when $y = 0$.

   $y(0) = 2 \cdot 0 + 5 = 5$, so our $y$–intercept is $(0, 5)$.

2. Look at the coefficient multiplying the $x$.

   *Every time we increase $x$ by one, $y$ increases by two, so our slope is $+2$.*

3. Plot the line with the given slope that goes through the intercept. We start at the point $(0, 5)$ and move over one in the $x$–direction, then up two in the $y$–direction. This gives the slope for our line, which we extend in both directions.

We will properly examine this last method later in this chapter!

Watch this video for help with the Examples above.
2.2. Solution Sets and Graphs of Linear Equations

Vocabulary

- The **coordinate plane** is a two-dimensional space defined by a horizontal number line (the \( x \)-axis) and a vertical number line (the \( y \)-axis). The **origin** is the point where these two lines meet. Four areas, or **quadrants**, are formed as shown in the diagram above.
- Each point on the coordinate plane has a set of **coordinates**, two numbers written as an **ordered pair** which describe how far along the \( x \)-axis and \( y \)-axis the point is. The \( x \)-**coordinate** is always written first, then the \( y \)-**coordinate**, in the form \((x, y)\).
- **Functions** are a way that we can relate one quantity to another. Functions can be plotted on the coordinate plane.
- A **Solution** to an equation is a number or set of coordinates that makes the equation true. All possible solutions to an equation is called the **solution set**.

Practice

For 1-3, make a table of values represented some of the solutions to the following equations and then graph them to visualize the solution set.

1. \( y = 2x + 7 \)
2. \( y = 0.7x - 4 \)
3. \( y = 6 - 1.25x \)

4. Determine if the ordered pair given is a solution to the equation given.
   - (a) Equation: \( x + y = 4 \)  Ordered Pair: \((2, -2)\).
   - (b) Equation: \( 2x = y + 3 \)  Ordered Pair: \((4, 5)\).
(c) Equation: \( y = 2 \). Ordered Pair: (-3, 2).
(d) Equation: \( x = 5 \). Ordered Pair: (0, 5).

5. “Think of a number. Multiply it by 20, divide the answer by 9, and then subtract seven from the result.”
   a. Make a table of values of some of the solutions and plot the function that represents this sentence.
   b. If you picked 0 as your starting number, what number would you end up with?
   c. To end up with 12, what number would you have to start out with?
   d. Write an equation to represent this function.

6. At the airport, you can change your money from dollars into euros. The service costs $5, and for every additional dollar you get 0.7 euros.
   a. Make a table of this and plot the function on a graph.
   b. Use your graph to determine how many euros you would get if you give the office $50.
   c. To get 35 euros, how many dollars would you have to pay?
   d. Write an equation to represent the solution set you graphed in part a.
   e. The exchange rate drops so that you can only get 0.5 euros per additional dollar. Now how many dollars do you have to pay for 35 euros? What is your new equation?

For 6-9 (we know...there are two number 6’s!), the graph below shows a conversion chart for converting between weight in kilograms and weight in pounds. Use it to convert the following measurements.

6. 4 kilograms into weight in pounds
7. 9 kilograms into weight in pounds
8. 12 pounds into weight in kilograms
9. 17 pounds into weight in kilograms

For 10-12, use the graph from problems 6-9 immediately above to answer the following questions.

10. An employee at a sporting goods store is packing 3-pound weights into a box that can hold 8 kilograms. How many weights can she place in the box?
11. After packing those weights, there is some extra space in the box that she wants to fill with one-pound weights. How many of those can she add?

12. After packing those, she realizes she misread the label and the box can actually hold 9 kilograms. How many more one-pound weights can she add?
2.3 Determining the Equation of a Line

Here you’ll learn how to write the equations of lines given their slope and y-intercept or two of their points. What if you were given the slope of a line and either its y-intercept or one of its points? Or what if you were given two of its points? How could you write the equation of that line? After completing this Concept, you’ll be able to write and graph equations from such information.

Watch This

Kahen Academ: YouTube Link: Equation of a Line

Try This

Another applet at http://www.cut-the-knot.org/Curriculum/Calculus/StraightLine.shtml lets you create multiple lines and see how they intersect. Each line is defined by two points; you can change the slope of a line by moving either of the points, or just drag the whole line around without changing its slope. To create another line, just click Duplicate and then drag one of the lines that are already there. You can also change the form the equation is written in. For example, change from "two points" to "slope-intercept".

Guidance

We saw in the last section that the solution set to a linear function is a set of points best represented on a graph. In this section, you will learn more about writing equations of lines and graphing equations of lines.

Write an Equation Given Slope and y–Intercept

You’ve already learned how to write an equation in slope–intercept form: simply start with the general equation for the slope-intercept form of a line, \( y = mx + b \), and then substitute the given values of \( m \) and \( b \) into the equation. For example, a line with a slope of 4 and a y–intercept of -3 would have the equation \( y = 4x - 3 \).

If you are given just the graph of a line, you can read off the slope and y–intercept from the graph and write the equation from there. For example, on the graph below you can see that the line rises by 1 unit as it moves 2 units to the right, so its slope is \( \frac{1}{2} \). Also, you can see that the y–intercept is -2, so the equation of the line is \( y = \frac{1}{2}x - 2 \).
2.3. Determining the Equation of a Line

Write an Equation Given the Slope and a Point

Often, we don’t know the value of the $y$-intercept, but we know the value of $y$ for a non-zero value of $x$. In this case, it’s often easier to write an equation of the line in point-slope form. An equation in point-slope form is written as $y - y_0 = m(x - x_0)$, where $m$ is the slope and $(x_0, y_0)$ is a point on the line.

**Example A**

A line has a slope of $\frac{3}{5}$, and the point (2, 6) is on the line. Write the equation of the line in point-slope form.

**Solution**

Start with the formula $y - y_0 = m(x - x_0)$.

Substitute $\frac{3}{5}$ for $m$, 2 for $x_0$ and 6 for $y_0$.

The equation in point-slope form is $y - 6 = \frac{3}{5}(x - 2)$.

Notice that the equation in point-slope form is not solved for $y$. If we did solve it for $y$, we’d have it in $y$-intercept form. To do that, we would just need to distribute the $\frac{3}{5}$ and add 6 to both sides. That means that the equation of this line in slope-intercept form is $y = \frac{3}{5}x - \frac{6}{5} + 6$, or simply $y = \frac{3}{5}x + \frac{24}{5}$.

**Write an Equation Given Two Points**

Point-slope form also comes in useful when we need to find an equation given just two points on a line.

For example, suppose we are told that the line passes through the points (-2, 3) and (5, 2). To find the equation of the line, we can start by finding the slope.

Starting with the slope formula, $m = \frac{y_2 - y_1}{x_2 - x_1}$, we plug in the $x-$ and $y-$values of the two points to get $m = \frac{2 - 3}{5 - (-2)} = \frac{-1}{7}$.

We can substitute that value of $m$ into the point-slope formula to get $y - y_0 = \frac{-1}{7}(x - x_0)$.

Now we just need to pick one of the two points to plug into the formula. Let’s use (5, 2); that gives us $y - 2 = \frac{-1}{7}(x - 5)$.

What if we’d picked the other point instead? Then we’d have ended up with the equation $y - 3 = \frac{-1}{7}(x + 2)$, which doesn’t look the same. That’s because there’s more than one way to write an equation for a given line in point-slope form. But let’s see what happens if we solve each of those equations for $y$.

Starting with $y - 2 = \frac{-1}{7}(x - 5)$, we distribute the $\frac{-1}{7}$ and add 2 to both sides. That gives us $y = -\frac{1}{7}x + \frac{5}{7} + 2$, or $y = -\frac{1}{7}x + \frac{19}{7}$.

On the other hand, if we start with $y - 3 = \frac{-1}{7}(x + 2)$, we need to distribute the $\frac{-1}{7}$ and add 3 to both sides. That
gives us \( y = -\frac{1}{7}x - \frac{2}{7} + 3 \), which also simplifies to \( y = -\frac{1}{7}x + \frac{19}{7} \).

So whichever point we choose to get an equation in point-slope form, the equation is still mathematically the same, and we can see this when we convert it to \( y \)-intercept form.

**Example B**

A line contains the points (3, 2) and (-2, 4). Write an equation for the line in point-slope form; then write an equation in \( y \)-intercept form.

**Solution**

Find the slope of the line: 
\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - 2}{-2 - 3} = -\frac{2}{5}
\]

Substitute in the value of the slope: \( y - y_0 = \frac{-2}{5}(x - x_0) \).

Substitute the point (3, 2) into the equation: 
\[
y - 2 = -\frac{2}{5}(x - 3).
\]

The equation in point-slope form is 
\[
y - 2 = -\frac{2}{5}(x - 3).
\]

To convert to \( y \)-intercept form, simply solve for \( y \):

\[
y - 2 = -\frac{2}{5}(x - 3) \rightarrow y - 2 = -\frac{2}{5}x + \frac{6}{5}
\]
\[
\rightarrow y = -\frac{2}{5}x + \frac{6}{5} + 2
\]
\[
\rightarrow y = -\frac{2}{5}x + \frac{31}{5}.
\]

The equation in \( y \)-intercept form is 
\[
y = -\frac{2}{5}x + \frac{31}{5}.
\]

**Graph an Equation in Point-Slope Form**

Another useful thing about point-slope form is that you can use it to graph an equation without having to convert it to slope-intercept form. From the equation \( y - y_0 = m(x - x_0) \), you can just read off the slope \( m \) and the point \((x_0, y_0)\). To draw the graph, all you have to do is plot the point, and then use the slope to figure out how many units up and over you should move to find another point on the line.

**Example C**

Make a graph of the line given by the equation \( y + 2 = \frac{2}{5}(x - 2) \).

**Solution**

To read off the right values, we need to rewrite the equation slightly: 
\[
y = \frac{2}{5}(x - 2).
\]

Now we see that point (2, -2) is on the line and that the slope is \( \frac{2}{5} \).

First plot point (2, -2) on the graph:
A slope of $\frac{2}{3}$ tells you that from that point you should move 2 units up and 3 units to the right and draw another point:

Now draw a line through the two points and extend it in both directions:
Watch this video for help with the Examples above.

CK-12 Foundation: Linear Equations

Vocabulary

- Often, we don’t know the value of the $y$–intercept, but we know the value of $y$ for a non-zero value of $x$. In this case, it’s often easier to write an equation of the line in **point-slope form**. An equation in point-slope form is written as $y - y_0 = m(x - x_0)$, where $m$ is the slope and $(x_0, y_0)$ is a point on the line.

Guided Practice

A line contains the points (1, -2) and (0, 0). Write an equation for the line in point-slope form; then write an equation in $y$–intercept form.

Solution

Find the slope of the line: $m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{-2 - 0}{1 - 0} = \frac{-2}{1} = -2$

Substitute in the value of the slope: $y - y_0 = -2(x - x_0)$.

Substitute the point (1, -2) into the equation: $y - (-2) = -2(x - 1)$.

The equation in point-slope form is $y + 2 = -2(x - 1)$.

To convert to $y$–intercept form, simply solve for $y$:

\[
y + 2 = -2(x - 1) \rightarrow y + 2 = -2x + 2 \\
\rightarrow y = -2x + 2 - 2 \\
\rightarrow y = -2x.
\]

The equation in $y$–intercept form is $y = -2x$.

Practice

Find the equation of each line in slope–intercept form.

1. The line has a slope of 7 and a $y$–intercept of -2.
2. The line has a slope of -5 and a $y$–intercept of 6.
3. The line has a slope of $-\frac{1}{4}$ and contains the point (4, -1).
4. The line contains points (3, 5) and (-3, 0).
5. The line contains points (10, 15) and (12, 20).

Write the equation of each line in slope-intercept form.
2.3. Determining the Equation of a Line

Find the equation of each linear function in slope–intercept form.

8. \( m = 5, f(0) = -3 \)
9. \( m = -7, f(2) = -1 \)
10. \( m = \frac{1}{3}, f(-1) = \frac{2}{3} \)
11. \( m = 4.2, f(-3) = 7.1 \)
12. \( f \left( \frac{1}{3} \right) = \frac{3}{4}, f(0) = \frac{5}{4} \)
13. \( f(1.5) = -3, f(-1) = 2 \)

Write the equation of each line in point-slope form.

14. The line has slope \(-\frac{1}{10}\) and goes through the point (10, 2).
15. The line has slope -75 and goes through the point (0, 125).
16. The line has slope 10 and goes through the point (8, -2).
17. The line goes through the points (-2, 3) and (-1, -2).
18. The line contains the points (10, 12) and (5, 25).
19. The line goes through the points (2, 3) and (0, 3).
20. The line has a slope of \(\frac{3}{4}\) and a \(y\)-intercept of -3.
21. The line has a slope of -6 and a \(y\)-intercept of 0.5.
22. \( m = -\frac{1}{5} \) and \( f(0) = 7 \)
23. \( m = -12 \) and \( f(-2) = 5 \)
24. \( f(-7) = 5 \) and \( f(3) = -4 \)
25. \( f(6) = 0 \) and \( f(0) = 6 \)
26. \( m = 3 \) and \( f(2) = -9 \)
27. \( m = -\frac{9}{5} \) and \( f(0) = 32 \)
28. \( f(1) = 5 \) and \( f(-4) = 5 \).
2.4 Forms of Linear Equations

Here you’ll learn how to write equations of lines in the standard form of \( ax + by = c \). You’ll also learn how to find the slope and \( y \)-intercept of lines written in standard form.

In this Concept, you will learn how to write equations in standard form.

Watch This

\[ \text{CK-12 Foundation: 0502S Standard Form of Linear Equations (H264)} \]

Try This

Now that you’ve worked with equations in all three basic forms, check out the Java applet at http://www.ronblond.com/M10/lineAP/index.html. You can use it to manipulate graphs of equations in all three forms, and see how the graphs change when you vary the terms of the equations.

Guidance

You’ve already encountered another useful form for writing linear equations: standard form. An equation in standard form is written \( ax + by = c \), where \( a, b, \) and \( c \) are all integers and \( a \) is positive. (Note that the \( b \) in the standard form is different than the \( b \) in the slope-intercept form.)

One useful thing about standard form is that it allows us to write equations for vertical lines, which we can’t do in slope-intercept form.

For example, let’s look at the line that passes through points (2, 6) and (2, 9). How would we find an equation for that line in slope-intercept form?

First we’d need to find the slope: \[ m = \frac{9-6}{2-2} = \frac{3}{0} \]. But that slope is undefined because we can’t divide by zero. And if we can’t find the slope, we can’t use point-slope form either.

If we just graph the line, we can see that \( x \) equals 2 no matter what \( y \) is. There’s no way to express that in slope-intercept or point-slope form, but in standard form we can just say that \( x + 0y = 2 \), or simply \( x = 2 \).

Converting to Standard Form

To convert an equation from another form to standard form, all you need to do is rewrite the equation so that all the variables are on one side of the equation and the coefficient of \( x \) is not negative.
Example A

Rewrite the following equations in standard form:

a) \( y = 5x - 7 \)

b) \( y - 2 = -3(x + 3) \)

c) \( y = \frac{2}{3}x + \frac{1}{2} \)

Solution

We need to rewrite each equation so that all the variables are on one side and the coefficient of \( x \) is not negative.

a) \( y = 5x - 7 \)

Subtract \( y \) from both sides to get \( 0 = 5x - y - 7. \)

Add 7 to both sides to get \( 7 = 5x - y. \)

Flip the equation around to put it in standard form: \( 5x - y = 7. \)

b) \( y - 2 = -3(x + 3) \)

Distribute the \(-3\) on the right-hand-side to get \( y - 2 = -3x - 9. \)

Add 3\( x \) to both sides to get \( y + 3x - 2 = -9. \)

Add 2 to both sides to get \( y + 3x = -7. \) Flip that around to get \( 3x + y = -7. \)

c) \( y = \frac{2}{3}x + \frac{1}{2} \)

Find the common denominator for all terms in the equation – in this case that would be 6.

Multiply all terms in the equation by 6: \( 6(y = \frac{2}{3}x + \frac{1}{2}) \Rightarrow 6y = 4x + 3. \)

Subtract 6\( y \) from both sides: \( 0 = 4x - 6y + 3. \)

Subtract 3 from both sides: \( -3 = 4x - 6y. \)

The equation in standard form is \( 4x - 6y = -3. \)

Graphing Equations in Standard Form

When an equation is in slope-intercept form or point-slope form, you can tell right away what the slope is. How do you find the slope when an equation is in standard form?

Well, you could rewrite the equation in slope-intercept form and read off the slope. But there’s an even easier way.

Let’s look at what happens when we rewrite an equation in standard form.

Starting with the equation \( ax + by = c \), we would subtract \( ax \) from both sides to get \( by = -ax + c. \) Then we would divide all terms by \( b \) and end up with \( y = -\frac{a}{b}x + \frac{c}{b}. \)

That means that the slope is \(-\frac{a}{b}\) and the \( y\)–intercept is \( \frac{c}{b}. \) So next time we look at an equation in standard form, we don’t have to rewrite it to find the slope; we know the slope is just \(-\frac{a}{b}\), where \( a \) and \( b \) are the coefficients of \( x \) and \( y \) in the equation.

Example B

Find the slope and the \( y\)–intercept of the following equations written in standard form.

a) \( 3x + 5y = 6 \)

b) \( 2x - 3y = -8 \)

c) \( x - 5y = 10 \)
2.4. Forms of Linear Equations

Solution

a) \(a = 3, b = 5,\) and \(c = 6,\) so the slope is \(-\frac{a}{b} = -\frac{3}{5},\) and the \(y-\)intercept is \(\frac{c}{b} = \frac{6}{5}.\)

b) \(a = 2, b = -3,\) and \(c = -8,\) so the slope is \(-\frac{a}{b} = \frac{2}{3},\) and the \(y-\)intercept is \(\frac{c}{b} = \frac{8}{3}.\)

c) \(a = 1, b = -5,\) and \(c = 10,\) so the slope is \(-\frac{a}{b} = \frac{1}{5},\) and the \(y-\)intercept is \(\frac{c}{b} = \frac{10}{5} = -2.\)

Once we’ve found the slope and \(y-\)intercept of an equation in standard form, we can graph it easily. But if we start with a graph, how do we find an equation of that line in standard form?

First, remember that we can also use the cover-up method to graph an equation in standard form, by finding the intercepts of the line. For example, let’s graph the line given by the equation \(3x - 2y = 6.\)

To find the \(x-\)intercept, cover up the \(y\) term (remember, the \(x-\)intercept is where \(y = 0)\):

\[
3x = 6 \Rightarrow x = 2
\]

The \(x-\)intercept is \((2, 0).\)

To find the \(y-\)intercept, cover up the \(x\) term (remember, the \(y-\)intercept is where \(x = 0)\):

\[
-2y = 6 \Rightarrow y = -3
\]

The \(y-\)intercept is \((0, -3).\)

We plot the intercepts and draw a line through them that extends in both directions:

Now we want to apply this process in reverse—to start with the graph of the line and write the equation of the line in standard form.
Example C

Find the equation of each line and write it in standard form.

a)

b)

c)
2.4. Forms of Linear Equations

**Solution**

a) We see that the $x-$intercept is $(3, 0) \Rightarrow x = 3$ and the $y-$intercept is $(0, -4) \Rightarrow y = -4$

We saw that in standard form $ax + by = c$: if we “cover up” the $y$ term, we get $ax = c$, and if we “cover up” the $x$ term, we get $by = c$.

So we need to find values for $a$ and $b$ so that we can plug in 3 for $x$ and -4 for $y$ and get the same value for $c$ in both cases. This is like finding the least common multiple of the $x-$ and $y-$intercepts.

In this case, we see that multiplying $x = 3$ by 4 and multiplying $y = -4$ by -3 gives the same result:

$$(x = 3) \times 4 \Rightarrow 4x = 12 \quad \text{and} \quad (y = -4) \times (-3) \Rightarrow -3y = 12$$

Therefore, $a = 4, b = -3$ and $c = 12$ and the equation in standard form is $4x - 3y = 12$.

b) We see that the $x-$intercept is $(3, 0) \Rightarrow x = 3$ and the $y-$intercept is $(0, 3) \Rightarrow y = 3$

The values of the intercept equations are already the same, so $a = 1, b = 1$ and $c = 3$. The equation in standard form is $x + y = 3$.

c) We see that the $x-$intercept is $(\frac{3}{2}, 0) \Rightarrow x = \frac{3}{2}$ and the $y-$intercept is $(0, 4) \Rightarrow y = 4$

Let’s multiply the $x-$intercept equation by 2 $\Rightarrow 2x = 3$

Then we see we can multiply the $x-$intercept again by 4 and the $y-$intercept by 3, so we end up with $8x = 12$ and $3y = 12$.

The equation in standard form is $8x + 3y = 12$.

Watch this video for help with the Examples above.

**MEDIA**

Click image to the left for use the URL below.

**Vocabulary**

- An equation in **standard form** is written \( ax + by = c \), where \( a, b, \) and \( c \) are all integers and \( a \) is positive. (Note that the \( b \) in the standard form is different than the \( b \) in the slope-intercept form.)

**Guided Practice**

**Find the slope and the \( y \)–intercept of the following equations written in standard form.**

a) \( 10x + 2y = 5 \)

b) \( 21x - 3y = -9 \)

**Solution:**

a) \( a = 10, b = 2, \) and \( c = 5, \) so the slope is \( \frac{a}{b} = -\frac{10}{2} = -5, \) and the \( y \)–intercept is \( \frac{c}{b} = \frac{5}{2} = 2.5 \).

b) \( a = 21, b = -3, \) and \( c = -9, \) so the slope is \( \frac{a}{b} = -\frac{21}{-3} = 7, \) and the \( y \)–intercept is \( \frac{c}{b} = \frac{-9}{-3} = 3 \).

**Practice**

For 1-6, rewrite the following equations in standard form.

1. \( y = 3x - 8 \)
2. \( y - 7 = -5(x - 12) \)
3. \( 2y = 6x + 9 \)
4. \( y = \frac{9}{3}x + \frac{1}{4} \)
5. \( y + \frac{3}{5} = \frac{2}{3}(x - 2) \)
6. \( 3y + 5 = 4(x - 9) \)

For 7-12, find the slope and \( y \)–intercept of the following lines.

7. \( 5x - 2y = 15 \)
8. \( 3x + 6y = 25 \)
9. \( x - 8y = 12 \)
10. \( 3x - 7y = 20 \)
11. \( 9x - 9y = 4 \)
12. \( 6x + y = 3 \)

For 13-14, find the equation of each line and write it in standard form.
15. Create your own graph of a vertical line. Write the equation for the graph.
Applications of Linear Functions

Introduction
In this lesson you will learn that linear relationships are often used to model real-life situations. To do this, two data values related to the real-life situation must be present in the problem. These two data values in context will give you the information necessary to create a graph and an equation to model the real-life situation. The graph and the equation will be more meaningful if the axis is labelled according to the items in the problem and if variables representing these items are used in the equation. When the data values have been represented graphically and the equation of the line has been determined, questions relating to the real-life situation can be presented and answered.

Objectives
The lesson objectives for Linear Function as a Model:

- Understanding how to determine the equation of a line that models a real-life situation
- Understanding the meaning of the slope of a line as it applies to real-life situations
- Understanding how to use the equation to answer problems related to the real-life situation

Introduction
In this lesson, you will learn that linear functions can be applied to real-life problems. When equations and graphs are used to model real-life situations, the domain of the graph is often $x \in \mathbb{N}$. However, it is often more convenient to sketch the graph as though $x \in \mathbb{R}$ instead of showing the function as a series of points in the plane.

Watch This
Khan Academy Basic Linear Function

Guidance
Joe’s Warehouse has banquet facilities to accommodate a maximum of 250 people. When the manager quotes a price for a banquet she is including the cost of renting the room plus the cost of the meal. A banquet for 70 people costs $1300. For 120 people, the price is $2200.

(a) Plot a graph of cost versus the number of people.
(b) From the graph, estimate the cost of a banquet for 150 people.
(c) Determine the slope of the line. What quantity does the slope of the line represent?
(d) Write an equation to model this real-life situation.
(a) On the $x$–axis is the number of people and on the $y$–axis is the cost of the banquet.

(b) The approximate cost of a banquet for 150 people is $2700.

(c) The two data points (70, 1300) and (120, 2200) will be used to calculate the slope of the line.

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ m = \frac{2200 - 1300}{120 - 70} \]
\[ m = \frac{900}{50} \]
\[ m = 18 \]

The slope represents the cost of the banquet for each person. The cost is $18 per person.

When a linear function is used to model the real life situation, the equation can be written in the form $y = mx + b$ or in the form $Ax + By + C = 0$.

(d) $y = mx + b$

\[ 1300 = 18(70) + b \]
\[ 1300 = 1260 + b \]
\[ 1300 - 1260 = 1260 - 1260 + b \]
\[ 40 = b \]

The $y$–intercept is (0, 40)

The equation to model the real-life situation is $y = 18x + 40$. The variables should be changed to match the labels on the axes. The equation that best models the situation is $c = 18n + 40$ where ’$c$’ represents the cost and ’$n$’ represents the number of people.

Example A

A cab company charges $2.00 for the first 0.6 miles and $0.50 for each additional 0.2 miles.

(a) Draw the graph of cost versus distance.

(b) Determine the equations that model this situation.

(c) What is the cost to travel 16 miles by cab?

This example will demonstrate a real-life situation that cannot be modeled with just one equation.
(a) On the $x$–axis is the distance in miles and on the $y$–axis is the cost in dollars. The first graph from $A$ to $B$ extends horizontally across the distance from 0 to 0.6 miles. The cost is constant at $2.00. The equation for this constant function is $y = 2.00$ or $c = 2.00$. The second graph from $B$ to $C$ and upward is not constant.

(b) The equation that models the second graph can be determined by using the data points $(0.6, 2.00)$ and $(1, 3.00)$. This second data point was found by locating a point on the graph that has exact coordinates.

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{3.00 - 2.00}{1 - 0.6} = \frac{1.00}{0.4} = 2.5
\]

Use the data points to calculate the slope.

\[
y - y_1 = m(x - x_1)
\]

Use the slope and one point to determine the equation.

\[
y - 2 = 2.5(x - 0.6)
y - 2 = 2.5x - 1.5
y - 2 + 2 = 2.5x - 1.5 + 2
y = 2.5x + 0.5
\]

Therefore, the equations that model this situation are:
2.5. Linear Function as a Model

\[
c = \begin{cases} 
2.00 & 0 < d \leq 0.6 \\
2.5d + 0.5 & d > 0.6 
\end{cases}
\]

(c) The cost to travel 16 miles in the cab is:

The distance is greater than 0.6 miles. The cost must be calculated using the equation \( c = 2.5d + 0.5 \). Substitute 16 in for \( d \).

\[
c = 2.5d + 0.5 \\
c = 2.5(16) + 0.5 \\
c = 2.5(16) + 0.5 \\
c = 40.50
\]

Example B

When a 40 gram mass was suspended from a coil spring, the length of the spring was 24 inches. When an 80 gram mass was suspended from the same coil spring, the length of the spring was 36 inches.

(a) Plot a graph of length versus mass.

(b) From the graph, estimate the length of the spring for a mass of 70 grams.

(c) Determine an equation that models this situation. Write the equation in slope-intercept form.

(d) Use the equation to determine the length of the spring for a mass of 60 grams.

(e) What is the \( y \)-intercept? What does the \( y \)-intercept represent?

(a) On the \( x \)-axis is the mass in grams and on the \( y \)-axis is the length of the spring in inches.

(b) The length of the coil spring for a mass of 70 grams is approximately 33 inches.

(c) The equation of the line can be determined by using the two data values (40, 24) and (80, 36).
\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]
\[ m = \frac{36 - 24}{80 - 40} \]
\[ m = \frac{12}{40} \]
\[ m = \frac{3}{10} \]

\[ y = mx + b \]
\[ 24 = \frac{3}{10} (40) + b \]
\[ 24 = \frac{3}{10} \times 40 + b \]
\[ 24 = 12 + b \]
\[ 24 - 12 = 12 - 12 + b \]
\[ 12 = b \]

The \( y \)-intercept is \((0, 12)\). The equation that models the situation is

\[ y = \frac{3}{10} x + 12 \]

\[ l = \frac{3}{10} m + 12 \]

where ’\( l \)' is the length of the spring in inches and ’\( m \)' is the mass in grams.

(d)

\[ l = \frac{3}{10} m + 12 \]

Use the equation and substitute 60 in for \( m \).

\[ l = \frac{3}{10} \times 60 + 12 \]
\[ l = \frac{3}{10} \times 60 + 12 \]
\[ l = 18 + 12 \]
\[ l = 30 \text{ inches} \]

(e) The \( y \)-intercept is \((0, 12)\). The \( y \)-intercept represents the length of the coil spring before a mass was suspended from it. The length of the coil spring was 12 inches.

**Example C**

Juan drove from his mother’s home to his sister’s home. After driving for 20 minutes he was 62 miles away from his sister’s home and after driving for 32 minutes he was only 38 miles away. The time driving and the distance away from his sister’s home form a linear relationship.

(a) What is the independent variable? What is the dependent variable?

(b) What are the two data values?

(c) Draw a graph to represent this problem. Label the axis appropriately.

(d) Write an equation expressing distance in terms of time driving.
(e) What is the slope and what is its meaning in this problem?
(f) What is the time-intercept and what does it represent?
(g) What is the distance-intercept and what does it represent?
(h) How far is Juan from his sister’s home after he has been driving for 35 minutes?

(a) The independent variable is the time driving. The dependent variable is the distance.
(b) The two data values are (20, 62) and (32, 38).
(c) On the $x$—axis is the time in minutes and on the $y$—axis is the distance in miles.

(d) (20, 62) and (32, 38) are the coordinates that will be used to calculate the slope of the line.

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{38 - 62}{32 - 20} = \frac{-24}{12} = -2
\]

\[
y = mx + b
\]

\[
62 = -2(20) + b \quad \Rightarrow \quad 62 = -40 + b \quad \Rightarrow \quad 62 + 40 = -40 + 40 + b \quad \Rightarrow \quad 102 = b \quad \text{The } y \text{— intercept is } (0, 102)
\]

\[
y = mx + b
\]
\[
y = -2x + 102
d = -2t + 102
\]

(e) The slope is $-2 = \frac{-2}{1} = \frac{-2\text{(miles)}}{1\text{(minute)}}$. The slope means that for each minute of driving, the distance that Juan has to drive to his sister’s home is reduced by 2 miles.

(f) The time-intercept is actually the $x$—intercept. This value is:
\( d = -2t + 102 \)  

Set \( d = 0 \) and solve for \( t \).

\[
0 = -2t + 102 \\
0 + 2t = -2t + 2t + 102 \\
2t = 102 \\
\frac{2t}{2} = \frac{102}{2} \\
2t = 51 \text{ minutes}
\]

The time-intercept is 51 minutes and this represents the time it took Juan to drive from his mother’s home to his sister’s home.

(g) The distance-intercept is the \( y \)–intercept. This value has been calculated as (0, 102). The distance-intercept represents the distance between his mother’s home and his sister’s home. The distance is 102 miles.

(h) 

\[
d = -2t + 102 \\
d = -2(35) + 102 \\
d = -70 + 102 \\
d = 32 \text{ miles}
\]

After driving for 35 minutes, Juan is 32 miles from his sister’s home.

**Guided Practice**

1. Some college students, who plan on becoming math teachers, decide to set up a tutoring service for high school math students. One student was charged $25 for 3 hours of tutoring. Another student was charged $55 for 7 hours of tutoring. The relationship between the cost and time is linear.

(a) What is the independent variable?

(b) What is the dependent variable?

(c) What are two data values for this relationship?

(d) Draw a graph of cost versus time.

(e) Determine an equation to model the situation.

(f) What is the significance of the slope?

(g) What is the cost-intercept? What does the cost-intercept represent?

2. A Glace Bay developer has produced a new handheld computer called the **Blueberry**. He sold 10 computers in one location for $1950 and 15 in another for $2850. The number of computers and cost forms a linear relationship.

(a) State the dependent and independent variables.

(b) Sketch a graph.

(c) Find an equation expressing cost in terms of the number of computers.

(d) State the slope of the line and tell what the slope means to the problem.

(e) State the cost-intercept and tell what it means to this problem.

(f) Using your equation, calculate the number of computers you could get for $6000.
3. Handy Andy sells one quart can of paint thinner for $7.65 and a two quart can for $13.95. Assume there is a linear relationship between the volume of paint thinner and the price.

(a) What is the independent variable?
(b) What is the dependent variable?
(c) Write two data values for this relationship.
(d) Draw a graph to represent this relationship.
(e) What is the slope of the line?
(f) What does the slope represent in this problem?
(g) Write an equation to model this problem.
(h) What is the cost-intercept?
(i) What does the cost-intercept represent in this problem?
(j) How much would you pay for 6 quarts of paint thinner?

Answers
1. (a) The cost for tutoring depends upon the amount of time. The independent variable is the time.
(b) The dependent variable is the cost.
(c) Two data values for this relationship are (3, 25) and (7, 55).
(d) On the $x$–axis is the time in hours and on the $y$–axis is the cost in dollars.

(e) Use the two data values (3, 25) and (7, 55) to calculate the slope of the line.

\[
m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{55 - 25}{7 - 3} = \frac{30}{4} = \frac{15}{2}
\]

Determine the $y$–intercept of the graph.
\[ y = mx + b \]

\[ 25 = \frac{15}{2} (3) + b \]  
Use the slope and one of the data values to determine the value of \( b \).

\[ 25 = \frac{45}{2} + b \]

\[ 25 \cdot \frac{2}{2} = \frac{45}{2} \cdot \frac{2}{2} + b \]

\[ \frac{50}{2} = \frac{45}{2} + \frac{b}{2} \]

\[ \frac{5}{2} = b \]

The \( y \)-intercept is \( \frac{5}{2} \)

The equation to model the relationship is \( y = \frac{15}{2} x + \frac{5}{2} \). To match the variables of the equation with the graph the equation is \( c = \frac{15}{2} t + \frac{5}{2} \). The relationship is cost in dollars versus time in hours. The equation could also be written as \( c = 7.50t + 2.50 \).

(f) The slope of \( \frac{15}{2} \) means that it costs \$15.00 for 2 hours of tutoring. If the slope is expressed as a decimal, it means that it costs \$7.50 for 1 hour of tutoring.

(g) The cost-intercept is the \( y \)-intercept. The \( y \)-intercept is \((0, 2.50)\). This value could represent the cost of having a scheduled time or the cost that must be paid for cancelling the appointment. In a problem like this, the \( y \)-intercept must be represented by a meaningful quantity for the problem.

2. (a) The number of dollars in sales from the computers depends upon the number of computers sold. The dependent variable is the dollars in sales and the independent variable is the number of computers sold.

(b) On the \( x \)-axis is the number of computers and on the \( y \)-axis is the cost of the computers.

![Graph of Cost vs. Number of Computers](image)

(c) Use the data values \((10, 1950)\) and \((15, 2850)\) to calculate the slope of the line.

\[ m = \frac{y_2 - y_1}{x_2 - x_1} \]

\[ m = \frac{2850 - 1950}{15 - 10} \]

\[ m = \frac{900}{5} \]

\[ m = 180 \]

Determine the \( y \)-intercept of the graph.
2.5. Linear Function as a Model

\[ y = mx + b \]

\[ 1950 = 180(10) + b \]
\[ 1950 = 1800 + b \]
\[ 1950 - 1800 = 1800 - 1800 + b \]
\[ 150 = b \]

The \( y \)-intercept is (0, 150).

The equation of the line that models the relationship is

\[ y = 180x + 150 \]

To make the equation match the variables of the graph the equation is

\[ c = 180n + 150 \]

(d) The slope is \( \frac{180}{1} \). This means that the cost of one computer is $180.00.

(e) The cost intercept is the \( y \)-intercept. The \( y \)-intercept is (0, 150). This could represent the cost of renting the location where the sales are being made or perhaps the salary for the sales person.

(f)

\[ c = 180n + 150 \]
\[ 6000 = 180n + 150 \]
\[ 6000 - 150 = 180n + 150 - 150 \]
\[ 5850 = 180n \]
\[ \frac{5850}{180} = \frac{180n}{180} \]
\[ n = \frac{32.5}{180} \]

With $6000 you could get 32 computers.

3. (a) The independent variable is the volume of paint thinner.

(b) The dependent variable is the cost of the paint thinner.

(c) Two data values are (1, 7.65) and (2, 13.95).

(d) On the \( x \)-axis is the volume in quarts and on the \( y \)-axis is the cost in dollars.

(e) Use the two data values (1, 7.65) and (2, 13.95) to calculate the slope of the line.
(f) The slope represents the cost of one quart of paint thinner. The cost is $6.30.

(g) 

\[ y = mx + b \]

\[ 7.65 = 6.30(1) + b \]

\[ 7.65 = 6.30 + b \]

\[ 7.65 - 6.30 = 6.30 - 6.30 + b \]

\[ 1.35 = b \]

The \( y \)-intercept is (0, 1.35). The equation to model the relationship is 

\[ y = 6.30x + 1.35 \]

The equation that matches the variables of the graph is \[ c = 6.30v + 1.35 \]

(h) The cost-intercept is (0, 1.35).

(i) This could represent the cost of the can that holds the paint thinner.

(j) 

\[ c = 6.30v + 1.35 \]

\[ c = 6.30(6) + 1.35 \]

\[ c = 37.80 + 1.35 \]

\[ c = 39.15 \]

The cost of 6 quarts of paint thinner is $39.15.

**Summary**

In this lesson you have learned that real-life problems can be represented by graphs and linear equations. The slope and the \( y \)-intercept both have significance that is reflected in the problem. The linear relationship can be modeled by a linear equation that reflects the variables of the graph. The equations can be written in either standard form or in \( y \)-intercept form.

Data that is discrete is represented using a linear graph instead of a set of plotted points.

**Problem Set**

**Completely answer the following problems...**

1. Players on the school soccer team are selling candles to raise money for an upcoming trip. Each player has 24 candles to sell. If a player sells 4 candles a profit of $30 is made. If he sells 12 candles a profit of $70 is made. The profit and the number of candles sold form a linear relation.

   a. State the dependent and the independent variables.
   b. What are the two data values for this relation?
   c. Draw a graph and label the axis.
d. Determine an equation to model this situation.
e. What is the slope and what does it mean in this problem?
f. Find the profit-intercept and explain what it represents.
g. Calculate the maximum profit that a player can make.
h. Write a suitable domain and range.
i. If a player makes a profit of $90, how many candles did he sell?
j. Is this data continuous or discrete? Justify your answer.

2. Jacob leaves his summer cottage and drives home. After driving for 5 hours, he is 112 km from home, and after 7 hours, he is 15 km from home. Assume that the distance from home and the number of hours driving form a linear relationship.

a. State the dependent and the independent variables.
b. What are the two data values for this relationship?
c. Represent this linear relationship graphically.
d. Determine the equation to model this situation.
e. What is the slope and what does it represent?
f. Find the distance-intercept and its real-life meaning in this problem.
g. How long did it take Jacob to drive from his summer cottage to home?
h. Write a suitable domain and range.
i. How far was Jacob from home after driving 4 hours?
j. How long had Jacob been driving when he was 209 km from home?

Summary

In this lesson you have learned to apply the equation of a linear relationship to a real-life situation. Two data values from the problem were used to determine the equation of the line. The slope of the line represented a meaningful quantity of the problem. The $y$–intercept also had significance for the real-life situation. The significance of the $y$–intercept must represent a quantity that is meaningful for the problem.
2.6 Graphing Linear Inequalities in Two Variables

Objective
To graph a linear inequality in two variables on the Cartesian plane.

Review Queue
Graph the following lines on the same set of axes.
1. \( y = \frac{1}{3}x - 2 \)
2. \( x - y = -6 \)
3. \( 2x + 5y = 15 \)

Solve the following inequalities. Graph the answer on a number line.
4. \( 4x - 5 \leq 11 \)
5. \( \frac{3}{5}x > -12 \)
6. \( -6x + 11 \geq -13 \)

Testing Solutions for Linear Inequalities in Two Variables

Objective
To determine if an ordered pair is a solution to a linear inequality in two variables.

Watch This
James Sousa: Ex: Determine if Ordered Pairs Satisfy a Linear Inequality

Guidance
A linear inequality is very similar to the equation of a line, but with an inequality sign. They can be written in one of the following ways:

\[ Ax + By < C \quad Ax + By > C \quad Ax + By \leq C \quad Ax + By \geq C \]

Notice that these inequalities are very similar to the standard form of a line. We can also write a linear inequality in slope-intercept form.

\[ y < mx + b \quad y > mx + b \quad y \leq mx + b \quad y \geq mx + b \]
In all of these general forms, the \( A, B, C, m, \) and \( b \) represent the exact same thing they did with lines.

An ordered pair, or point, is a solution to a linear inequality if it makes the inequality true when the values are substituted in for \( x \) and \( y \).

**Example A**
Which ordered pair is a solution to \( 4x - y > -12 \)?

a) (6, -5)
b) (-3, 0)
c) (-5, 4)

**Solution:** Plug in each point to see if they make the inequality true.

a) 
\[
4(6) - (-5) > -12
\]

\[
24 + 5 > -12
\]

\[
29 > -12
\]

b) 
\[
4(-3) - 0 > -12
\]

\[
-12 \not> -12
\]

c) 
\[
4(-5) - 4 > -12
\]

\[
-20 - 4 > -12
\]

\[
-24 \not> -12
\]

Of the three points, a) is the only one where the inequality holds true. b) is not true because the inequality sign is only “greater than,” not “greater than or equal to.”

**Vocabulary**

**Linear Inequality**
An inequality, usually in two variables, of the form \( Ax + By < C, Ax + By > C, Ax + By \leq C, \) or \( Ax + By \geq C. \)

**Solution**
An ordered pair that satisfies a given inequality.

**Guided Practice**
1. Which inequality is (-7, 1) a solution for?
   a) \( y < 2x - 1 \)
   b) \( 4x - 3y \geq 9 \)
   c) \( y > -4 \)
2. List three possible solutions for \( 5x - y \leq 3. \)

**Answers**
1. Plug (-7, 1) in to each equation. With c), only use the \( y \) value.
a) 
\[1 < 2(-7) - 1\]
\[1 < -15\]

b) 
\[4(-7) - 3(1) \geq 9\]
\[-28 - 3 \geq 9\]
\[-31 \ngeq 9\]

c) \[1 > -4\]

\((-7, 1)\) is only a solution to \(y > -4\).

2. To find possible solutions, plug in values to the inequality. There are infinitely many solutions. Here are three:
\((-1, 0), (-4, 3),\) and \((1, 6)\).

\[
\begin{align*}
5(-1) - 0 & \leq 3 \\
-5 & \leq 3 \\
5(-4) - 3 & \leq 3 \\
-17 & \leq 3 \\
5(1) - 6 & \leq 3 \\
-1 & \leq 3
\end{align*}
\]

Problem Set

Using the four inequalities below, determine which point is a solution for each one. There may be more than one correct answer. If the answer is none, write none of these.

A) \(y \leq \frac{2}{3}x - 5\)
B) \(5x + 4y > 20\)
C) \(x - y \geq -5\)
D) \(y > -4x + 1\)

1. \((9, -1)\)
2. \((0, 0)\)
3. \((-1, 6)\)
4. \((-3, -10)\)

Determine which inequality each point is a solution for. There may be more than one correct answer. If the answer is none, write none of these.

A) \((-5, 1)\)
B) \((4, 2)\)
C) \((-12, -7)\)
D) \((8, -9)\)

5. \(2x - 3y > 8\)
6. \(y \leq -x - 4\)
7. \(y \geq 6x + 7\)
8. \(8x + 3y < -3\)
9. Is \((-6, -8)\) a solution to \(y < \frac{1}{2}x - 6\)?
10. Is \((10, 1)\) a solution to \(y \geq -7x + 1\)?
For problems 11 and 12, find three solutions for each inequality.

11. \(5x - y > 12\)
12. \(y \leq -2x + 9\)

Graphing Inequalities in Two Variables

Objective
To graph a linear inequality on the Cartesian plane.

Watch This

Khan Academy: Graphing linear inequalities in two variables 2

Guidance

Graphing linear inequalities is very similar to graphing lines. First, you need to change the inequality into slope-intercept form. At this point, we will have a couple of differences. If the inequality is in the form \(y < mx + b\) or \(y > mx + b\), the line will be dotted or dashed because it is not a part of the solution. If the line is in the form \(y \leq mx + b\) or \(y \geq mx + b\), the line will be solid to indicate that it is included in the solution.

The second difference is the shading. Because these are inequalities, not just the line is the solution. Depending on the sign, there will be shading above or below the line. If the inequality is in the form \(y < mx + b\) or \(y \leq mx + b\), the shading will be below the line, in reference to the \(y\)-axis.

If the inequality is in the form \(y > mx + b\) or \(y \geq mx + b\), the shading will be above the line.
Example A

Graph \(4x - 2y < 10\).

**Solution:** First, change the inequality into slope-intercept form. Remember, that if you have to divide or multiply by a negative number, you must flip the inequality sign.

\[
4x - 2y < 10 \\
-2y < -4x + 10 \quad \text{Flip the inequality sign because we divided by -2.} \\
y > 2x - 5
\]

Now, graph the inequality as if it was a line. Plot \(y = 2x - 5\) like in the *Graphing Lines in Slope-Intercept Form* concept. However, the line will be dashed because of the “greater than” sign.

Now, we need to determine the shading. You can use one of two methods to do this. The first way is to use the graphs and forms from above. The equation, in slope-intercept form, matches up with the purple dashed line and shading. Therefore, we should shade above the dashed blue line.
The alternate method would be to test a couple of points to see if they work. If a point is true, then the shading is over that side of the line. If we pick (-5, 0), the inequality yields $-20 < 10$, which tells us that our shading is correct.

**Example B**

Graph $y \leq -\frac{2}{3}x + 6$.

**Solution:** This inequality is already in slope-intercept form. So, graph the line, which will be solid, and then determine the shading. Looking at the example graphs above, this inequality should look like the red inequality, so shade below the line.

Test a point to make sure our shading is correct. An easy point in the shaded region is (0, 0). Plugging this into the inequality, we get $0 \leq 6$, which is true.

**Example C**

Graph $y \geq 4$.

**Solution:** Treat this inequality like you are graphing a horizontal line. We will draw a solid line at $y = 4$ and then shade above because of the “≥” sign.
Example D

Determine the linear inequality that is graphed below.

Solution: Find the equation of the line portion just like you did in the Find the Equation of a Line in Slope-Intercept Form concept. The given points on the line are (0, 8) and (6, 2) (from the points drawn on the graph). This means that the $y$–intercept is (0, 8). Then, using slope triangles we fall 6 and run 6 to get to (6, 2). This means the slope is $-\frac{6}{6}$ or -1. Because we have a dotted line and the shading is above, our sign will be the $>$ sign. Putting it all together, the equation of our linear inequality is $y > -x + 8$.

*When finding the equation of an inequality, like above, it is easiest to find the equation in slope-intercept form. To determine which inequality sign to use, look at the shading along the $y$–axis. If the shaded $y$–values get larger, the line will be in the form $y > mx + b$ or $y \geq mx + b$. If they get smaller, the line will be in the form $y < mx + b$ or $y \leq mx + b$.

**Guided Practice**

1. Graph $3x - 4y > 20$.
2. Graph $x < -1$.
3. What is the equation of the linear inequality?
2.6. Graphing Linear Inequalities in Two Variables

Answers

1. First, change the inequality into slope-intercept form.

\[3x - 4y > 20\]
\[-4y > -3x + 20\]
\[y < \frac{3}{4}x - 5\]

Now, we need to determine the type of line and shading. Because the sign is “<,” the line will be dashed and we will shade below.

Test a point in the shaded region to make sure we are correct. If we test (6, -6) in the original inequality, we get 42 > 20, which is true.

2. To graph this line on the \(x - y\) plane, recall that all vertical lines have the form \(x = a\). Therefore, we will have a vertical dashed line at -1. Then, the shading will be to the left of the dashed line because that is where \(x\) will be less than the value of the line.
3. Looking at the line, the \( y\)–intercept is \((0, 1)\). Using a slope triangle, to count down to the next point, we would fall 4, and run backward 1. This means that the slope is \(\frac{4}{1} = 4\). The line is solid and the shading is above, so we will use the \(\geq\) sign. Our inequality is \(y \geq 4x + 1\).

**Problem Set**

Graph the following inequalities.

1. \(y > x - 5\)
2. \(3x - 2y \geq 4\)
3. \(y < -3x + 8\)
4. \(x + 4y \leq 16\)
5. \(y < -2\)
6. \(y < -\frac{1}{2}x - 3\)
7. \(x \geq 6\)
8. \(8x + 4y \geq -20\)
9. \(-4x + y \leq 7\)
10. \(5x - 3y \geq -24\)
11. \(y > 5x\)
12. \(y \leq 0\)

Determine the equation of each linear inequality below.
2.6. Graphing Linear Inequalities in Two Variables

13.

14.

15.
16.
2.7 Solving Linear Systems by Graphing

Objective
Review the concept of the solution to a linear system in the context of a graph.

Review Queue
1. Graph the equation \( y = \frac{1}{2}x + 3 \).
2. Write the equation \( 4x - 3y = 6 \) in slope intercept form.
3. Solve for \( x \) in \( 7x - 3y = 26 \), given that \( y = 3 \).

NOTE: This section contains 3 Parts with THREE different Problem Sets.

Part 1: Checking a Solution for a Linear System / Problem Set #1
Part 2: Solving Systems with One Solution Using Graphing / Problem Set #2
Part 3: Solving Systems with No or Infinitely Many Solutions with Graphing / Problem Set #3

Part 1: Checking a Solution for a Linear System

Objective
Determine whether an ordered pair is a solution to a given system of linear equations.

Watch This

James Sousa: Ex: Identify the Solution to a System of Equations Given a Graph, Then Verify

Guidance
A system of linear equations consists of the equations of two lines. The solution to a system of linear equations is the point which lies on both lines. In other words, the solution is the point where the two lines intersect. To verify whether a point is a solution to a system or not, we will either determine whether it is the point of intersection of two lines on a graph (Example A) or we will determine whether or not the point lies on both lines algebraically (Example B).

Example A
Is the point (5, -2) the solution of the system of linear equations shown in the graph below?
Solution: Yes, the lines intersect at the point (5, -2) so it is the solution to the system.

Example B
Is the point (-3, 4) the solution to the system given below?

\[
\begin{align*}
2x - 3y &= -18 \\
\quad x + 2y &= 6
\end{align*}
\]

Solution: No, (-3, 4) is not the solution. If we replace the \(x\) and \(y\) in each equation with -3 and 4 respectively, only the first equation is true. The point is not on the second line; therefore it is not the solution to the system.

Guided Practice

1. Is the point (-2, 1) the solution to the system shown below?
2. Verify algebraically that (6, -1) is the solution to the system shown below.

\[
\begin{align*}
3x - 4y &= 22 \\
2x + 5y &= 7
\end{align*}
\]

3. Explain why the point (3, 7) is the solution to the system:

\[
\begin{align*}
y &= 7 \\
x &= 3
\end{align*}
\]

**Answers**

1. No, (-2, 1) is not the solution. The solution is where the two lines intersect which is the point (-3, 1).

2. By replacing \(x\) and \(y\) in both equations with 6 and -1 respectively (shown below), we can verify that the point (6, -1) satisfies both equations and thus lies on both lines.

\[
\begin{align*}
3(6) - 4(-1) &= 18 + 4 = 22 \\
2(6) + 5(-1) &= 12 - 5 = 7
\end{align*}
\]

3. The horizontal line is the line containing all points where the \(y\)-coordinate is 7. The vertical line is the line containing all points where the \(x\)--coordinate is 3. Thus, the point (3, 7) lies on both lines and is the solution to the system.

**Problem Set #1**

Match the solutions with their systems.

1. \((1, 2)\)
a.

b.
2. (2, 1)
3. (-1, 2)
4. (-1, -2)
2.7. Solving Linear Systems by Graphing

Determine whether each ordered pair represents the solution to the given system.

5. 

\[4x + 3y = 12\]
\[5x + 2y = 1; \ (-3, 8)\]

6. 

\[3x - y = 17\]
\[2x + 3y = 5; \ (5, -2)\]
7. 
\[7x - 9y = 7\]
\[x + y = 1; \ (1, 0)\]

8. 
\[x + y = -4\]
\[x - y = 4; \ (5, -9)\]

9. 
\[x = 11\]
\[y = 10; \ (11, 10)\]

10. 
\[x + 3y = 0\]
\[y = -5; \ (15, -5)\]

11. Describe the solution to a system of linear equations.
12. Can you think of why a linear system of two equations would not have a unique solution?

**Part 2: Solving Systems with One Solution Using Graphing**

**Objective**

Graph lines to identify the unique solution to a system of linear equations.

**Watch This**

James Sousa: Ex 1: Solve a System of Equations by Graphing

**Guidance**

In this lesson we will be using various techniques to graph the pairs of lines in systems of linear equations to identify the point of intersection or the solution to the system. It is important to use graph paper and a straightedge to graph the lines accurately. Also, you are encouraged to check your answer algebraically as described in the previous lesson.

**Example A**

Graph and solve the system:

\[y = -x + 1\]
\[y = \frac{1}{2}x - 2\]
Solution:
Since both of these equations are written in slope intercept form, we can graph them easily by plotting the $y$–intercept point and using the slope to locate additional points on each line.

The equation, $y = -x + 1$, graphed in blue, has $y$–intercept 1 and slope $-\frac{1}{1}$.

The equation, $y = \frac{1}{2}x - 2$, graphed in red, has $y$–intercept -2 and slope $\frac{1}{2}$.

Now that both lines have been graphed, the intersection is observed to be the point (2, -1).

Check this solution algebraically by substituting the point into both equations.

Equation 1: $y = -x + 1$, making the substitution gives: $(-1) = (-2) + 1$. ✓

Equation 2: $y = \frac{1}{2}x - 2$, making the substitution gives: $-1 = \frac{1}{2}(2) - 2$. ✓

(2, -1) is the solution to the system.

Example B
Graph and solve the system:

$3x + 2y = 6$
$y = -\frac{1}{2}x - 1$

Solution: This example is very similar to the first example. The only difference is that equation 1 is not in slope intercept form. We can either solve for $y$ to put it in slope intercept form or we can use the intercepts to graph the equation. To review using intercepts to graph lines, we will use the latter method.

Recall that the $x$–intercept can be found by replacing $y$ with zero and solving for $x$:
\[ 3x + 2(0) = 6 \]
\[ 3x = 6 \]
\[ x = 2 \]

Similarly, the \( y \)-intercept is found by replacing \( x \) with zero and solving for \( y \):

\[ 3(0) + 2y = 6 \]
\[ 2y = 6 \]
\[ y = 3 \]

We have two points, \((2, 0)\) and \((0, 3)\) to plot and graph this line. Equation 2 can be graphed using the \( y \)-intercept and slope as shown in Example A.

Now that both lines are graphed we observe that their intersection is the point \((4, -3)\).

Finally, check this solution by substituting it into each of the two equations.

**Equation 1:** \[3x + 2y = 6; 3(4) + 2(-3) = 12 - 6 = 6 \] \( \checkmark \)

**Equation 2:** \( y = -\frac{1}{2}x - 1; -3 = -\frac{1}{2}(4) - 1 \) \( \checkmark \)

**Example C**

In this example we will use technology to solve the system:

\[ 2x - 3y = 10 \]
\[ y = -\frac{2}{3}x + 4 \]
2.7. Solving Linear Systems by Graphing

This process may vary somewhat based on the technology you use. All directions here can be applied to the TI-83 or 84 (plus, silver, etc) calculators.

**Solution:** The first step is to graph these equations on the calculator. The first equation must be rearranged into slope intercept form to put in the calculator.

\[
2x - 3y = 10 \\
-3y = -2x + 10 \\
y = \frac{-2x + 10}{-3} \\
y = \frac{2}{3}x - \frac{10}{3}
\]

The graph obtained using the calculator should look like this:

The solution does not lie on the “grid” and is therefore difficult to observe visually. With technology we can calculate the intersection. If you have a TI-83 or 84, use the CALC menu, select INTERSECT. Then select each line by pressing ENTER on each one. The calculator will give you a “guess.” Press ENTER one more time and the calculator will then calculate the intersection of (5.5, .333...). We can also write this point as \((\frac{11}{2}, \frac{1}{3})\). Check the solution algebraically.

**Equation 1:**  \[2x - 3y = 10; 2 \left( \frac{11}{2} \right) - 3 \left( \frac{1}{3} \right) = 11 - 1 = 10 \quad \checkmark\]

**Equation 2:**  \[y = -\frac{2}{3}x + 4; -\frac{2}{3} \left( \frac{11}{2} \right) + 4 = -\frac{11}{3} + \frac{12}{3} = \frac{1}{3} \quad \checkmark\]

Guided Practice

Solve the following systems by graphing. Use technology for problem 3.
1.

\[ y = 3x - 4 \]
\[ y = 2 \]

2.

\[ 2x - y = -4 \]
\[ 2x + 3y = -12 \]

3.

\[ 5x + y = 10 \]
\[ y = \frac{2}{3}x - 7 \]

**Answers**

1.

The first line is in slope intercept form and can be graphed accordingly.
The second line is a horizontal line through \((0, 2)\).
The graph of the two equations is shown below. From this graph the solution appears to be \((2, 2)\).
Checking this solution in each equation verifies that it is indeed correct.

**Equation 1:** \[ 2 = 3(2) - 4 \] \(\checkmark\)

**Equation 2:** \[ 2 = 2 \] \(\checkmark\)
Neither of these equations is in slope intercept form. The easiest way to graph them is to find their intercepts as shown in Example B.

Equation 1: Let \( y = 0 \) to find the \( x \)–intercept.

\[
\begin{align*}
2x - y &= -4 \\
2x - 0 &= -4 \\
x &= -2
\end{align*}
\]

Now let \( x = 0 \), to find the \( y \)–intercept.

\[
\begin{align*}
2x - y &= -4 \\
2(0) - y &= -4 \\
y &= 4
\end{align*}
\]

Now we can use (-2, 0) and (0, 4) to graph the line as shown in the diagram. Using the same process, the intercepts for the second line can be found to be (-6, 0) and (0, -4).

Now the solution to the system can be observed to be (-3, -2). This solution can be verified algebraically as shown in the first problem.

3.
The first equation here must be rearranged to be $y = -5x + 10$ before it can be entered into the calculator. The second equation can be entered as is.

Using the calculate menu on the calculator the solution is $(3, -5)$.

Remember to verify this solution algebraically as a way to check your work.

**Problem Set #2**

Match the system of linear equations to its graph and state the solution.

1. 

   \[
   \begin{align*}
   3x + 2y &= -2 \\
   x - y &= -4
   \end{align*}
   \]
2.7. Solving Linear Systems by Graphing

b.

c.
2.

\[2x - y = 6\]
\[2x + 3y = 6\]
3.

\[ 2x - 5y = -5 \]
\[ x + 5y = 5 \]
4.

\[ y = 5x - 5 \]
\[ y = -x + 1 \]
2.7. Solving Linear Systems by Graphing

b.

c.
Solve the following linear systems by graphing. Use graph paper and a straightedge to insure accuracy. You are encouraged to verify your answer algebraically.

5. 
\[ y = -\frac{2}{5}x + 1 \]
\[ y = \frac{3}{5}x - 4 \]

6. 
\[ y = -\frac{2}{3}x + 4 \]
\[ y = 3x - 7 \]

7. 
\[ y = -2x + 1 \]
\[ x - y = -4 \]

8. 
\[ 3x + 4y = 12 \]
\[ x - 4y = 4 \]

9. 
\[ 7x - 2y = -4 \]
\[ y = -5 \]
10. \[ \begin{align*} x - 2y &= -8 \\ x &= -3 \end{align*} \]

Solve the following linear systems by graphing using technology. Solutions should be rounded to the nearest hundredth as necessary.

11. \[ \begin{align*} y &= \frac{3}{7}x + 11 \\ y &= -\frac{13}{2}x - 5 \end{align*} \]

12. \[ \begin{align*} y &= 0.95x - 8.3 \\ 2x + 9y &= 23 \end{align*} \]

13. \[ \begin{align*} 15x - y &= 22 \\ 3x + 8y &= 15 \end{align*} \]

Use the following information to complete exercises 14-17.

Clara and her brother, Carl, are at the beach for vacation. They want to rent bikes to ride up and down the boardwalk. One rental shop, Bargain Bikes, advertises rates of $5 plus $1.50 per hour. A second shop, Frugal Wheels, advertises a rate of $6 plus $1.25 an hour.

14. How much does it cost to rent a bike for one hour from each shop? How about 10 hours?
15. Write equations to represent the cost of renting a bike from each shop. Let \( x \) represent the number of hours and \( y \) represent the total cost.
16. Solve your system to figure out when the cost is the same.
17. Clara and Carl want to rent the bikes for about 3 hours. Which shop should they use?

**Part 3: Solving Systems with No or Infinitely Many Solutions Using Graphing**

**Objective**

Determine whether a system has a unique solution or not based on its graph. If no unique solution exists, determine whether there is no solution or infinitely many solutions.

**Guidance**

So far we have looked at linear systems of equations in which the lines always intersected in one, unique point. What happens if this is not the case? What could the graph of the two lines look like? In Examples A and B below we will explore the two possibilities.

**Example A**

Graph the system:

\[ \begin{align*} y &= 2x - 5 \\ y &= 2x + 4 \end{align*} \]
Solution:
In this example both lines have the same slope but different \( y \)–intercepts. When graphed, they are parallel lines and never intersect. This system has no solution. Another way to say this is to say that it is inconsistent.

Example B
Graph the system:

\[
\begin{align*}
2x - 3y &= 6 \\
-4x + 6y &= -12
\end{align*}
\]
Solution:
In this example both lines have the same slope and \( y \)-intercept. This is more apparent when the equations are written in slope intercept form:

\[
y = \frac{2}{3} x - 2 \quad \text{and} \quad y = \frac{2}{3} x - 2
\]

When we graph them, they are one line, coincident, meaning they have all points in common. This means that there are an infinite number of solutions to the system. Because this system has at least one solution it is considered to be consistent.

Consistent systems are systems which have at least one solution. If the system has exactly one, unique solution then it is independent. All of the systems we solved in the last section were independent. If the system has infinite solutions, like the system in Example B, then it is called dependent.

Example C
Classify the following system:

\[
10x - 2y = 10 \\
y = 5x - 5
\]

Solution:
Rearranging the first equation into slope intercept form we get \( y = 5x - 5 \), which is exactly the same as the second equation. This means that they are the same line. Therefore the system is consistent and dependent.

Guided Practice
Classify the following systems as consistent, inconsistent, independent or dependent. You may do this with or without graphing them. You do not need to find the unique solution if there is one.
1.

\[
\begin{align*}
5x - y &= 15 \\
x + 5y &= 15
\end{align*}
\]

2.

\[
\begin{align*}
9x - 12y &= -24 \\
-3x + 4y &= 8
\end{align*}
\]

3.

\[
\begin{align*}
6x + 8y &= 12 \\
-3x - 4y &= 10
\end{align*}
\]

**Answers**

1. The first step is to rearrange both equations into slope intercept form so that we can compare these attributes.

\[
\begin{align*}
5x - y &= 15 \rightarrow y = 5x - 15 \\
x + 5y &= 15 \rightarrow y = \frac{1}{5}x + 3
\end{align*}
\]

The slopes are not the same so the lines are neither parallel nor coincident. Therefore, the lines must intersect in one point. The system is consistent and independent.

2. Again, rearrange the equations into slope intercept form:

\[
\begin{align*}
9x - 12y &= -24 \rightarrow y = \frac{3}{4}x + 2 \\
-3x + 4y &= 8 \rightarrow y = \frac{3}{4}x + 2
\end{align*}
\]

Now, we can see that both the slope and the \(y\)--intercept are the same and therefore the lines are coincident. The system is consistent and dependent.

3. The equations can be rewritten as follows:

\[
\begin{align*}
6x + 8y &= 12 \rightarrow y = -\frac{3}{4}x + \frac{3}{2} \\
-3x - 4y &= 10 \rightarrow y = -\frac{3}{4}x - \frac{5}{2}
\end{align*}
\]

In this system the lines have the same slope but different \(y\)--intercepts so they are parallel lines. Therefore the system is inconsistent. There is no solution.

**Vocabulary**

**Parallel**

Two or more lines in the same plane that never intersect. They have the same slope and different \(y\)--intercepts.
Coincident
Lines which have all points in common. They are line which “coincide” with one another or are the same line.

Consistent
Describes a system with at least one solution.

Inconsistent
Describes a system with no solution.

Dependent
Describes a consistent system with infinite solutions.

Independent
Describes a consistent system with exactly one solution.

Problem Set #3
Describe the systems graphed below both algebraically (consistent, inconsistent, dependent, independent) and geometrically (intersecting lines, parallel lines, coincident lines).
Classify the following systems as consistent, inconsistent, independent or dependent. You may do this with or without graphing them. You do not need to find the unique solution if there is one.

4.

\begin{align*}
4x - y &= 8 \\
y &= 4x + 3
\end{align*}

5.

\begin{align*}
5x + y &= 10 \\
y &= 5x + 10
\end{align*}
6. 
\[ 2x - 2y = 11 \]
\[ y = x + 13 \]

7. 
\[ -7x + 3y = -21 \]
\[ 14x - 6y = 42 \]

8. 
\[ y = -\frac{3}{5}x + 1 \]
\[ 3x + 5y = 5 \]

9. 
\[ 6x - y = 18 \]
\[ y = \frac{1}{6}x + 3 \]

In problems 10-12 you will be writing your own systems. Your equations should be in standard form, \( Ax + By = C \). Try to make them look different even if they are the same equation.

10. Write a system which is consistent and independent.
11. Write a system which is consistent and dependent.
12. Write a system which is inconsistent.
2.8 Solving Systems Using Substitution

Here you’ll learn how to use substitution to solve systems of linear equations in two variables. You’ll then solve real-world problems involving such systems.

What if you were given a system of linear equations like \( x - y = 7 \) and \( 3x - 4y = -3 \)? How could you substitute one equation into the other to solve for the variables? After completing this Concept, you’ll be able to solve a system of linear equations by substitution.

Try This

For lots more practice solving linear systems, check out this web page: http://www.algebra.com/algebra/homework/coordinate/practice-linear-system.epl

After clicking to see the solution to a problem, you can click the back button and then click Try Another Practice Linear System to see another problem.

Guidance

In this lesson, we’ll learn to solve a system of two equations using the method of substitution.

Solving Linear Systems Using Substitution of Variable Expressions

Let’s look again at the problem about Peter and Nadia racing.

Peter and Nadia like to race each other. Peter can run at a speed of 5 feet per second and Nadia can run at a speed of 6 feet per second. To be a good sport, Nadia likes to give Peter a head start of 20 feet. How long does Nadia take to catch up with Peter? At what distance from the start does Nadia catch up with Peter?

In that example we came up with two equations:

Nadia’s equation: \( d = 6t \)

Peter’s equation: \( d = 5t + 20 \)

Each equation produced its own line on a graph, and to solve the system we found the point at which the lines intersected—the point where the values for \( d \) and \( t \) satisfied both relationships. When the values for \( d \) and \( t \) are equal, that means that Peter and Nadia are at the same place at the same time.

But there’s a faster way than graphing to solve this system of equations. Since we want the value of \( d \) to be the same in both equations, we could just set the two right-hand sides of the equations equal to each other to solve for \( t \). That is, if \( d = 6t \) and \( d = 5t + 20 \), and the two \( d \)’s are equal to each other, then by the transitive property we have \( 6t = 5t + 20 \). We can solve this for \( t \):

\[
\begin{align*}
6t &= 5t + 20 \\
\text{subtract } 5t \text{ from both sides:} & \\
t &= 20 \\
\text{substitute this value for } t \text{ into Nadia’s equation:} & \\
d &= 6 \cdot 20 = 120
\end{align*}
\]

Even if the equations weren’t so obvious, we could use simple algebraic manipulation to find an expression for one variable in terms of the other. If we rearrange Peter’s equation to isolate \( t \):
2.8. Solving Systems Using Substitution

\[ d = 5t + 20 \]  \hspace{1cm} \text{subtract 20 from both sides:} \]
\[ d - 20 = 5t \]  \hspace{1cm} \text{divide by 5:} \]
\[ \frac{d - 20}{5} = t \]

We can now substitute this expression for \( t \) into Nadia’s equation \((d = 6t)\) to solve:

\[ d = 6 \left( \frac{d - 20}{5} \right) \]  \hspace{1cm} \text{multiply both sides by 5:} \]
\[ 5d = 6(d - 20) \]  \hspace{1cm} \text{distribute the 6:} \]
\[ 5d = 6d - 120 \]  \hspace{1cm} \text{subtract 6d from both sides:} \]
\[ -d = -120 \]  \hspace{1cm} \text{divide by \(-1: \)} \]
\[ d = 120 \]  \hspace{1cm} \text{substitute value for d into our expression for t:} \]
\[ t = \frac{120 - 20}{5} = \frac{100}{5} = 20 \]

So we find that Nadia and Peter meet 20 seconds after they start racing, at a distance of 120 feet away.

The method we just used is called the **Substitution Method**. In this lesson you’ll learn several techniques for isolating variables in a system of equations, and for using those expressions to solve systems of equations that describe situations like this one.

**Example A**

Let’s look at an example where the equations are written in **standard form**.

*Solve the system*

\[ 2x + 3y = 6 \]
\[ -4x + y = 2 \]

Again, we start by looking to isolate one variable in either equation. If you look at the second equation, you should see that the coefficient of \( y \) is 1. So the easiest way to start is to use this equation to solve for \( y \).

Solve the second equation for \( y \):

\[ -4x + y = 2 \]  \hspace{1cm} \text{add 4x to both sides:} \]
\[ y = 2 + 4x \]

Substitute this expression into the first equation:

\[ 2x + 3(2 + 4x) = 6 \]  \hspace{1cm} \text{distribute the 3:} \]
\[ 2x + 6 + 12x = 6 \]  \hspace{1cm} \text{collect like terms:} \]
\[ 14x + 6 = 6 \]  \hspace{1cm} \text{subtract 6 from both sides:} \]
\[ 14x = 0 \]  \hspace{1cm} \text{and hence:} \]
\[ x = 0 \]
Substitute back into our expression for $y$:

$$y = 2 + 4 \cdot 0 = 2$$

As you can see, we end up with the same solution $(x = 0, y = 2)$ that we found when we graphed these functions back in Lesson 7.1. So long as you are careful with the algebra, the substitution method can be a very efficient way to solve systems.

Next, let’s look at a more complicated example. Here, the values of $x$ and $y$ we end up with aren’t whole numbers, so they would be difficult to read off a graph!

**Example B**

*Solve the system*

\[
\begin{align*}
2x + 3y &= 3 \\
2x - 3y &= -1
\end{align*}
\]

Again, we start by looking to isolate one variable in either equation. In this case it doesn’t matter which equation we use—all the variables look about equally easy to solve for.

So let’s solve the first equation for $x$:

\[
\begin{align*}
2x + 3y &= 3 \\
2x &= 3 - 3y \\
x &= \frac{1}{2}(3 - 3y)
\end{align*}
\]

Substitute this expression into the second equation:

\[
\begin{align*}
2 \cdot \frac{1}{2}(3 - 3y) - 3y &= -1 \\
3 - 3y - 3y &= -1 \\
3 - 6y &= -1 \\
-6y &= -4 \\
y &= \frac{2}{3}
\end{align*}
\]

Substitute into the expression we got for $x$:

\[
\begin{align*}
x &= \frac{1}{2} \left( 3 - \frac{2}{3} \right) \\
x &= \frac{1}{2}
\end{align*}
\]
2.8. Solving Systems Using Substitution

So our solution is \( x = \frac{1}{2}, y = \frac{2}{3} \). You can see how the graphical solution \((\frac{1}{2}, \frac{2}{3})\) might have been difficult to read accurately off a graph!

**Solving Real-World Problems Using Linear Systems**

Simultaneous equations can help us solve many real-world problems. We may be considering a purchase—for example, trying to decide whether it’s cheaper to buy an item online where you pay shipping or at the store where you do not. Or you may wish to join a CD music club, but aren’t sure if you would really save any money by buying a new CD every month in that way. Or you might be considering two different phone contracts. Let’s look at an example of that now.

**Example C**

Anne is trying to choose between two phone plans. The first plan, with Vendafone, costs $20 per month, with calls costing an additional 25 cents per minute. The second company, Sellnet, charges $40 per month, but calls cost only 8 cents per minute. Which should she choose?

You should see that Anne’s choice will depend upon how many minutes of calls she expects to use each month. We start by writing two equations for the cost in dollars in terms of the minutes used. Since the number of minutes is the independent variable, it will be our \( x \). Cost is dependent on minutes – the cost per month is the dependent variable and will be assigned \( y \).

*For Vendafone:* \( y = 0.25x + 20 \)

*For Sellnet:* \( y = 0.08x + 40 \)

By writing the equations in slope-intercept form \((y = mx + b)\), you can sketch a graph to visualize the situation:

The line for Vendafone has an intercept of 20 and a slope of 0.25. The Sellnet line has an intercept of 40 and a slope of 0.08 (which is roughly a third of the Vendafone line’s slope). In order to help Anne decide which to choose, we’ll find where the two lines cross, by solving the two equations as a system.

Since equation 1 gives us an expression for \( y(0.25x + 20) \), we can substitute this expression directly into equation 2:

\[
\begin{align*}
0.25x + 20 &= 0.08x + 40 \\
0.17x &= 20 \\
x &= 117.65 \text{ minutes}
\end{align*}
\]

So if Anne uses 117.65 minutes a month (although she can’t really do exactly that, because phone plans only count whole numbers of minutes), the phone plans will cost the same. Now we need to look at the graph to see which
plan is better if she uses more minutes than that, and which plan is better if she uses fewer. You can see that the Vendafone plan costs more when she uses more minutes, and the Sellnet plan costs more with fewer minutes.

So, if Anne will use 117 minutes or less every month she should choose Vendafone. If she plans on using 118 or more minutes she should choose Sellnet.

Watch this video for help with the Examples above.

CK-12 Foundation: Linear Systems by Substitution

Vocabulary

- Solving linear systems by substitution means to solve for one variable in one equation, and then to substitute it into the other equation, solving for the other variable.

Guided Practice

Solve the system

\[
8x + 10y = 2 \\
4x - 15y = -19
\]

Solution:

Again, we start by looking to isolate one variable in either equation. In this case it doesn’t matter which equation we use—all the variables look about equally easy to solve for.

So let’s solve the first equation for \(x\):

\[
8x + 10y = 2 \\
8x = 2 - 10y \\
x = \frac{1}{8}(2 - 10y)
\]

Substitute this expression into the second equation:
4 \cdot \frac{1}{8}(2 - 10y) - 15y = -19
\frac{1}{2}(2 - 10y) - 15y = -19
1 - 5y - 15y = -19
1 - 20y = -19
-20y = -20
y = 1

Simplify the fraction:
Distribute the fraction and re-write terms:
Collect like terms:
Subtract 1 from both sides:
Divide by -20:

Substitute into the expression we got for x:

\[ x = \frac{1}{8}(2 - 10y) \]
\[ x = \frac{1}{8}(2 - 10(1)) \]
\[ x = \frac{1}{8}(2 - 10) \]
\[ x = \frac{1}{8}(-8) \]
\[ x = -1 \]

Substitute the y-value into the x equation:
Simplify:

So our solution is \( x = -1, y = 1 \).

Practice

1. Solve the system:
   \[ x + 2y = 9 \]
   \[ 3x + 5y = 20 \]

2. Solve the system:
   \[ x - 3y = 10 \]
   \[ 2x + y = 13 \]

3. Solve the system:
   \[ 2x + 0.5y = -10 \]
   \[ x - y = -10 \]

4. Solve the system:
   \[ 2x + 0.5y = 3 \]
   \[ x + 2y = 8.5 \]

5. Solve the system:
   \[ 3x + 5y = -1 \]
   \[ x + 2y = -1 \]
6. Solve the system:

\[3x + 5y = -3\]
\[x + 2y = \frac{-4}{3}\]

7. Solve the system:

\[x - y = \frac{-12}{5}\]
\[2x + 5y = -2\]

8. Of the two non-right angles in a right angled triangle, one measures twice as many degrees as the other. What are the angles?

9. The sum of two numbers is 70. They differ by 11. What are the numbers?

10. A number plus half of another number equals 6; twice the first number minus three times the second number equals 4. What are the numbers?

11. A rectangular field is enclosed by a fence on three sides and a wall on the fourth side. The total length of the fence is 320 yards. If the field has a total perimeter of 400 yards, what are the dimensions of the field?

12. A ray cuts a line forming two angles. The difference between the two angles is 18°. What does each angle measure?

13. Jason is five years older than Becky, and the sum of their ages is 23. What are their ages?
2.9 Solving Linear Systems by Elimination

Learning Objectives

- Solve a linear system of equations using elimination by addition.
- Solve a linear system of equations using elimination by subtraction.
- Solve a linear system of equations by multiplication and then addition or subtraction.
- Compare methods for solving linear systems.
- Solve real-world problems using linear systems by any method.

Introduction

In this lesson, we’ll see how to use simple addition and subtraction to simplify our system of equations to a single equation involving a single variable. Because we go from two unknowns \((x\) and \(y\)) to a single unknown (either \(x\) or \(y\)), this method is often referred to by solving by elimination. We eliminate one variable in order to make our equations solvable! To illustrate this idea, let’s look at the simple example of buying apples and bananas.

**Example 1**

*If one apple plus one banana costs $1.25 and one apple plus 2 bananas costs $2.00, how much does one banana cost? One apple?*

It shouldn’t take too long to discover that each banana costs $0.75. After all, the second purchase just contains 1 more banana than the first, and costs $0.75 more, so that one banana must cost $0.75.

Here’s what we get when we describe this situation with algebra:

\[
\begin{align*}
a + b &= 1.25 \\
a + 2b &= 2.00
\end{align*}
\]

Now we can subtract the number of apples and bananas in the first equation from the number in the second equation, and also subtract the cost in the first equation from the cost in the second equation, to get the difference in cost that corresponds to the difference in items purchased.

\[
(a + 2b) - (a + b) = 2.00 - 1.25 \rightarrow b = 0.75
\]

That gives us the cost of one banana. To find out how much one apple costs, we subtract $0.75 from the total cost of one apple and one banana.

\[
a + 0.75 = 1.25 \rightarrow a = 1.25 - 0.75 \rightarrow a = 0.50
\]

So an apple costs 50 cents.

To solve systems using addition and subtraction, we’ll be using exactly this idea – by looking at the sum or difference of the two equations we can determine a value for one of the unknowns.
Watch This

http://www.youtube.com/watch?v=ova8GSmPV4o  James Sousa: Solving Systems of Equations by Elimination

Solving Linear Systems Using Addition of Equations

Often considered the easiest and most powerful method of solving systems of equations, the addition (or elimination) method lets us combine two equations in such a way that the resulting equation has only one variable. We can then use simple algebra to solve for that variable. Then, if we need to, we can substitute the value we get for that variable back into either one of the original equations to solve for the other variable.

Example 2

Solve this system by addition:

\[
\begin{align*}
3x + 2y &= 11 \\
5x - 2y &= 13
\end{align*}
\]

Solution

We will add everything on the left of the equals sign from both equations, and this will be equal to the sum of everything on the right:

\[
(3x + 2y) + (5x - 2y) = 11 + 13 \rightarrow 8x = 24 \rightarrow x = 3
\]

A simpler way to visualize this is to keep the equations as they appear above, and to add them together vertically, going down the columns. However, just like when you add units, tens and hundreds, you MUST be sure to keep the \(x\)'s and \(y\)'s in their own columns. You may also wish to use terms like “0\(y\)” as a placeholder!

\[
\begin{align*}
3x + 2y &= 11 \\
+ (5x - 2y) &= 13 \\
8x + 0y &= 24
\end{align*}
\]

Again we get \(8x = 24\), or \(x = 3\). To find a value for \(y\), we simply substitute our value for \(x\) back in.

Substitute \(x = 3\) into the second equation:

\[
\begin{align*}
5 \cdot 3 - 2y &= 13 \\
-2y &= -2 \\
y &= 1
\end{align*}
\]

since \(5 \times 3 = 15\), we subtract 15 from both sides:

divide by \(-2\) to get:
The reason this method worked is that the $y$—coefficients of the two equations were opposites of each other: 2 and -2. Because they were opposites, they canceled each other out when we added the two equations together, so our final equation had no $y$—term in it and we could just solve it for $x$.

In a little while we’ll see how to use the addition method when the coefficients are not opposites, but for now let’s look at another example where they are.

**Example 3**

Andrew is paddling his canoe down a fast-moving river. Paddling downstream he travels at 7 miles per hour, relative to the river bank. Paddling upstream, he moves slower, traveling at 1.5 miles per hour. If he paddles equally hard in both directions, how fast is the current? How fast would Andrew travel in calm water?

**Solution**

First we convert our problem into equations. We have two unknowns to solve for, so we’ll call the speed that Andrew paddles at $x$, and the speed of the river $y$. When traveling downstream, Andrew speed is boosted by the river current, so his total speed is his paddling speed plus the speed of the river $(x + y)$. Traveling upstream, the river is working against him, so his total speed is his paddling speed minus the speed of the river $(x - y)$.

Downstream Equation: $x + y = 7$

Upstream Equation: $x - y = 1.5$

Next we’ll eliminate one of the variables. If you look at the two equations, you can see that the coefficient of $y$ is +1 in the first equation and -1 in the second. Clearly $(+1) + (-1) = 0$, so this is the variable we will eliminate. To do this we simply add equation 1 to equation 2. We must be careful to collect like terms, and make sure that everything on the left of the equals sign stays on the left, and everything on the right stays on the right:

$$(x + y) + (x - y) = 7 + 1.5 \Rightarrow 2x = 8.5 \Rightarrow x = 4.25$$

Or, using the column method we used in example 2:

$$\begin{align*}
    x + y &= 7 \\
    + x - y &= 1.5 \\
    \hline
    2x + 0y &= 8.5
\end{align*}$$

Again we get $2x = 8.5$, or $x = 4.25$. To find a corresponding value for $y$, we plug our value for $x$ into either equation and isolate our unknown. In this example, we’ll plug it into the first equation:

$$4.25 + y = 7 \quad \text{subtract 4.25 from both sides:}$$

$$y = 2.75$$

Andrew paddles at 4.25 miles per hour. The river moves at 2.75 miles per hour.

**Solving Linear Systems Using Subtraction of Equations**

Another, very similar method for solving systems is subtraction. When the $x$— or $y$—coefficients in both equations are the same (including the sign) instead of being opposites, you can subtract one equation from the other.

If you look again at Example 3, you can see that the coefficient for $x$ in both equations is +1. Instead of adding the two equations together to get rid of the $y$'s, you could have subtracted to get rid of the $x$'s:


\[(x+y) - (x-y) = 7 - 1.5 \Rightarrow 2y = 5.5 \Rightarrow y = 2.75\]

or...

\[x + y = 7\]

\[- (x - y) = -1.5\]

\[0x + 2y = 5.5\]

So again we get \(y = 2.75\), and we can plug that back in to determine \(x\).

The method of subtraction is just as straightforward as addition, so long as you remember the following:

- Always put the equation you are subtracting in parentheses, and distribute the negative.
- Don’t forget to **subtract** the numbers on the right-hand side.
- Always remember that subtracting a negative is the same as adding a positive.

**Example 4**

*Peter examines the coins in the fountain at the mall. He counts 107 coins, all of which are either pennies or nickels. The total value of the coins is $3.47. How many of each coin did he see?*

**Solution**

We have 2 types of coins, so let’s call the number of pennies \(x\) and the number of nickels \(y\). The total value of all the pennies is just \(x\), since they are worth 1¢ each. The total value of the nickels is 5\(y\). We are given two key pieces of information to make our equations: the number of coins and their value in cents.

\[
\begin{align*}
\text{# of coins equation :} & \quad x + y = 107 & \text{(number of pennies) + (number of nickels)} \\
\text{value equation :} & \quad x + 5y = 347 & \text{pennies are worth 1¢, nickels are worth 5¢.}
\end{align*}
\]

We’ll jump straight to subtracting the two equations:

\[x + y = 107\]

\[- (x + 5y) = -347\]

\[- 4y = -240\]

\[y = 60\]

Substituting this value back into the first equation:

\[x + 60 = 107\]

**subtract 60 from both sides:**

\[x = 47\]

So Peter saw 47 pennies (worth 47 cents) and 60 nickels (worth $3.00) making a total of $3.47.

**Solving Linear Systems Using Multiplication**

So far, we’ve seen that the elimination method works well when the coefficient of one variable happens to be the same (or opposite) in the two equations. But what if the two equations don’t have any coefficients the same?
It turns out that we can still use the elimination method; we just have to make one of the coefficients match. We can accomplish this by multiplying one or both of the equations by a constant.

Here’s a quick review of how to do that. Consider the following questions:

1. If 10 apples cost $5, how much would 30 apples cost?
2. If 3 bananas plus 2 carrots cost $4, how much would 6 bananas plus 4 carrots cost?

If you look at the first equation, it should be obvious that each apple costs $0.50. So 30 apples should cost $15.00. The second equation is trickier; it isn’t obvious what the individual price for either bananas or carrots is. Yet we know that the answer to question 2 is $8.00. How?

If we look again at question 1, we see that we can write an equation: $10a = 5$ ($a$ being the cost of 1 apple). So to find the cost of 30 apples, we could solve for $a$ and then multiply by 30—but we could also just multiply both sides of the equation by 3. We would get $30a = 15$, and that tells us that 30 apples cost $15.

And we can do the same thing with the second question. The equation for this situation is $3b + 2c = 4$, and we can see that we need to solve for $(6b + 4c)$, which is simply 2 times $(3b + 2c)$! So algebraically, we are simply multiplying the entire equation by 2:

\[ 2(3b + 2c) = 2 \cdot 4 \rightarrow 6b + 4c = 8 \]

So when we multiply an equation, all we are doing is multiplying every term in the equation by a fixed amount.

**Solving a Linear System by Multiplying One Equation**

If we can multiply every term in an equation by a fixed number (a scalar), that means we can use the addition method on a whole new set of linear systems. We can manipulate the equations in a system to ensure that the coefficients of one of the variables match.

This is easiest to do when the coefficient as a variable in one equation is a multiple of the coefficient in the other equation.

**Example 5**

* Solve the system:

\[
\begin{align*}
7x + 4y &= 17 \\
5x - 2y &= 11
\end{align*}
\]

**Solution**

You can easily see that if we multiply the second equation by 2, the coefficients of $y$ will be +4 and -4, allowing us to solve the system by addition:

*2 times equation 2:*
\[
10x - 4y = 22 \\
+ (7x + 4y) = 17 \\
\hline 
17x = 34
\]

\textit{divide by 17 to get}: \quad x = 2

Now simply substitute this value for \( x \) back into equation 1:

\[
7 \cdot 2 + 4y = 17 \\
4y = 3 \\
y = 0.75
\]

\textbf{Example 6}

Anne is rowing her boat along a river. Rowing downstream, it takes her 2 minutes to cover 400 yards. Rowing upstream, it takes her 8 minutes to travel the same 400 yards. If she was rowing equally hard in both directions, calculate, in yards per minute, the speed of the river and the speed Anne would travel in calm water.

\textbf{Solution}

Step one: first we convert our problem into equations. We know that \textit{distance traveled} is equal to \textit{speed} \times \textit{time}. We have two unknowns, so we’ll call the speed of the river \( x \), and the speed that Anne rows at \( y \). When traveling downstream, her total speed is her rowing speed plus the speed of the river, or \((x + y)\). Going upstream, her speed is hindered by the speed of the river, so her speed upstream is \((x - y)\).

\textit{Downstream Equation:} \quad 2(x + y) = 400
\textit{Upstream Equation:} \quad 8(x - y) = 400

Distributing gives us the following system:

\[
2x + 2y = 400 \\
8x - 8y = 400
\]

Right now, we can’t use the method of elimination because none of the coefficients match. But if we multiplied the top equation by 4, the coefficients of \( y \) would be +8 and -8. Let’s do that:

\[
8x + 8y = 1,600 \\
\hline 
+ (8x - 8y) = 400 \\
16x = 2,000
\]

Now we divide by 16 to obtain \( x = 125 \).

Substitute this value back into the first equation:

\[
2(125 + y) = 400 \\
125 + y = 200 \\
y = 75
\]

\textit{divide both sides by 2}:

\textit{subtract 125 from both sides}: 

\textit{divide both sides by 2}:

\textit{subtract 125 from both sides}: 

\textit{divide both sides by 2}:

\textit{subtract 125 from both sides}:
Anne rows at 125 yards per minute, and the river flows at 75 yards per minute.

**Solving a Linear System by Multiplying Both Equations**

So what do we do if none of the coefficients match and none of them are simple multiples of each other? We do the same thing we do when we’re adding fractions whose denominators aren’t simple multiples of each other. Remember that when we add fractions, we have to find a lowest common denominator—that is, the lowest common multiple of the two denominators—and sometimes we have to rewrite not just one, but both fractions to get them to have a common denominator. Similarly, sometimes we have to multiply both equations by different constants in order to get one of the coefficients to match.

**Example 7**

Andrew and Anne both use the I-Haul truck rental company to move their belongings from home to the dorm rooms on the University of Chicago campus. I-Haul has a charge per day and an additional charge per mile. Andrew travels from San Diego, California, a distance of 2060 miles in five days. Anne travels 880 miles from Norfolk, Virginia, and it takes her three days. If Anne pays $840 and Andrew pays $1845, what does I-Haul charge

a) per day?

b) per mile traveled?

**Solution**

First, we’ll set up our equations. Again we have 2 unknowns: the daily rate (we’ll call this $x$), and the per-mile rate (we’ll call this $y$).

Anne’s equation: $3x + 880y = 840$

Andrew’s Equation: $5x + 2060y = 1845$

We can’t just multiply a single equation by an integer number in order to arrive at matching coefficients. But if we look at the coefficients of $x$ (as they are easier to deal with than the coefficients of $y$), we see that they both have a common multiple of 15 (in fact 15 is the lowest common multiple). So we can multiply both equations.

Multiply the top equation by 5:

$$15x + 4400y = 4200$$

Multiply the lower equation by 3:

$$15x + 6180y = 5535$$

Subtract:

$$
\begin{align*}
15x + 4400y & = 4200 \\
- (15x + 6180y) & = -5535 \\
\hline
- 1780y & = -1335
\end{align*}
$$

Divide by $-1780$: $y = 0.75$

Substitute this back into the top equation:
3x + 880(0.75) = 840  
3x = 180  
x = 60  

I-Haul charges $60 per day plus $0.75 per mile.

Review Questions

1. Solve the system:

   \[ 3x + 4y = 2.5 \]
   \[ 5x - 4y = 25.5 \]

2. Solve the system:

   \[ 5x + 7y = -31 \]
   \[ 5x - 9y = 17 \]

3. Solve the system:

   \[ 3y - 4x = -33 \]
   \[ 5x - 3y = 40.5 \]

4. Nadia and Peter visit the candy store. Nadia buys three candy bars and four fruit roll-ups for $2.84. Peter also buys three candy bars, but can only afford one additional fruit roll-up. His purchase costs $1.79. What is the cost of a candy bar and a fruit roll-up individually?

5. A small plane flies from Los Angeles to Denver with a tail wind (the wind blows in the same direction as the plane) and an air-traffic controller reads its ground-speed (speed measured relative to the ground) at 275 miles per hour. Another, identical plane, moving in the opposite direction has a ground-speed of 227 miles per hour. Assuming both planes are flying with identical air-speeds, calculate the speed of the wind.

6. An airport taxi firm charges a pick-up fee, plus an additional per-mile fee for any rides taken. If a 12-mile journey costs $14.29 and a 17-mile journey costs $19.91, calculate:
   a. the pick-up fee
   b. the per-mile rate
   c. the cost of a seven mile trip

7. Calls from a call-box are charged per minute at one rate for the first five minutes, then a different rate for each additional minute. If a 7-minute call costs $4.25 and a 12-minute call costs $5.50, find each rate.

8. A plumber and a builder were employed to fit a new bath, each working a different number of hours. The plumber earns $35 per hour, and the builder earns $28 per hour. Together they were paid $330.75, but the plumber earned $106.75 more than the builder. How many hours did each work?

9. Paul has a part time job selling computers at a local electronics store. He earns a fixed hourly wage, but can earn a bonus by selling warranties for the computers he sells. He works 20 hours per week. In his first week, he sold eight warranties and earned $220. In his second week, he managed to sell 13 warranties and earned $280. What is Paul’s hourly rate, and how much extra does he get for selling each warranty?

Solve the following systems using multiplication.
2.9. Solving Linear Systems by Elimination

10.  
\begin{align*}
5x - 10y &= 15 \\
3x - 2y &= 3
\end{align*}

11.  
\begin{align*}
5x - y &= 10 \\
3x - 2y &= -1
\end{align*}

12.  
\begin{align*}
5x + 7y &= 15 \\
7x - 3y &= 5
\end{align*}

13.  
\begin{align*}
9x + 5y &= 9 \\
12x + 8y &= 12.8
\end{align*}

14.  
\begin{align*}
4x - 3y &= 1 \\
3x - 4y &= 4
\end{align*}

15.  
\begin{align*}
7x - 3y &= -3 \\
6x + 4y &= 3
\end{align*}
### Table 2.4:

<table>
<thead>
<tr>
<th>Method:</th>
<th>Best used when you...</th>
<th>Advantages:</th>
<th>Comment:</th>
</tr>
</thead>
<tbody>
<tr>
<td>Graphing</td>
<td>...don’t need an accurate answer.</td>
<td>Often easier to see number and quality of intersections on a graph. With a graphing calculator, it can be the fastest method since you don’t have to do any computation.</td>
<td>Can lead to imprecise answers with non-integer solutions.</td>
</tr>
<tr>
<td>Substitution</td>
<td>...have an explicit equation for one variable (e.g. ( y = 14x + 2 ))</td>
<td>Works on all systems. Reduces the system to one variable, making it easier to solve.</td>
<td>You are not often given explicit functions in systems problems, so you may have to do extra work to get one of the equations into that form.</td>
</tr>
<tr>
<td>Elimination by Addition or Subtraction</td>
<td>...have matching coefficients for one variable in both equations.</td>
<td>Easy to combine equations to eliminate one variable. Quick to solve.</td>
<td>It is not very likely that a given system will have matching coefficients.</td>
</tr>
<tr>
<td>Elimination by Multiplication and then Addition and Subtraction</td>
<td>...do not have any variables defined explicitly or any matching coefficients.</td>
<td>Works on all systems. Makes it possible to combine equations to eliminate one variable.</td>
<td>Often more algebraic manipulation is needed to prepare the equations.</td>
</tr>
</tbody>
</table>
The table above is only a guide. You might prefer to use the graphical method for every system in order to better understand what is happening, or you might prefer to use the multiplication method even when a substitution would work just as well.

**Example A**

Two angles are **complementary** when the sum of their angles is 90°. Angles A and B are complementary angles, and twice the measure of angle A is 9° more than three times the measure of angle B. Find the measure of each angle.

**Solution**

First we write out our 2 equations. We will use \( x \) to be the measure of angle A and \( y \) to be the measure of angle B. We get the following system:

\[
\begin{align*}
x + y &= 90 \\
2x &= 3y + 9
\end{align*}
\]

First, we’ll solve this system with the graphical method. For this, we need to convert the two equations to \( y = mx + b \) form:

\[
\begin{align*}
x + y &= 90 \quad \Rightarrow \quad y = -x + 90 \\
2x &= 3y + 9 \quad \Rightarrow \quad y = \frac{2}{3}x - 3
\end{align*}
\]

The first line has a slope of -1 and a \( y \)-intercept of 90, and the second line has a slope of \( \frac{2}{3} \) and a \( y \)-intercept of -3. The graph looks like this:

In the graph, it appears that the lines cross at around \( x = 55, y = 35 \), but it is difficult to tell exactly! Graphing by hand is not the best method in this case!

**Example B**

In this example, we’ll try solving by substitution. Let’s look again at the system:
\[ x + y = 90 \\
2x = 3y + 9 \]

We’ve already seen that we can start by solving either equation for \( y \), so let’s start with the first one:

\[ y = 90 - x \]

Substitute into the second equation:

\[
\begin{align*}
2x &= 3(90 - x) + 9 \\
&= 270 - 3x + 9 \\
5x &= 270 + 9 = 279 \\
x &= 55.8^\circ
\end{align*}
\]

Substitute back into our expression for \( y \):

\[ y = 90 - 55.8 = 34.2^\circ \]

**Angle A measures 55.8°; angle B measures 34.2°.**

**Example C**

Finally, in this example, we’ll try solving by elimination (with multiplication):

Rearrange equation one to standard form:

\[ x + y = 90 \quad \Rightarrow \quad 2x + 2y = 180 \]

Multiply equation two by 2:

\[
\begin{align*}
2x &= 3y + 9 \\
&\Rightarrow \quad 2x - 3y = 9
\end{align*}
\]

Subtract:

\[
\begin{align*}
2x + 2y &= 180 \\
- (2x - 3y) &= -9 \\
\hline 
5y &= 171
\end{align*}
\]

Divide by 5 to obtain \( y = 34.2^\circ \)
2.10. Comparing Methods for Solving Linear Systems

Substitute this value into the very first equation:

\[ x + 34.2 = 90 \]
\[ x = 55.8^\circ \]

**Angle A measures** 55.8°; **angle B measures** 34.2°.

Even though this system looked ideal for substitution, the method of multiplication worked well too. Once the equations were rearranged properly, the solution was quick to find. You’ll need to decide yourself which method to use in each case you see from now on. Try to master all the techniques, and recognize which one will be most efficient for each system you are asked to solve.

Watch this video for help with the Examples above.


**Vocabulary**

- A **linear system of equations** is a set of equations that must be solved together to find the one solution that fits them both.

- Solving linear systems **by substitution** means to solve for one variable in one equation, and then to substitute it into the other equation, solving for the other variable.

- The purpose of the **elimination method** to solve a system is to cancel, or eliminate, a variable by either adding or subtracting the two equations. Sometimes the equations must be multiplied by scalars first, in order to cancel out a variable.

**Guided Practice**

Solve the system \[ \begin{cases} 5s + 2t = 6 \\ 9s + 2t = 22 \end{cases} \]

**Solution:**

Since these equations are both written in standard form, and both have the term 2t in them, we will use elimination by subtracting. This will cause the t terms to cancel out and we will be left with one variable, s, which we can then isolate.
\[
\begin{align*}
5s + 2t &= 6 \\
-(9s + 2t &= 22) \\
-4s + 0t &= -16 \\
-4s &= -16 \\
s &= 4 \\
\end{align*}
\]

\[
\begin{align*}
5(4) + 2t &= 6 \\
20 + 2t &= 6 \\
2t &= -14 \\
t &= -7 \\
\end{align*}
\]

The solution is \((4, -7)\).

**Practice**

Solve the following systems using any method.

1.

\[
\begin{align*}
x &= 3y \\
x - 2y &= -3 \\
\end{align*}
\]

2.

\[
\begin{align*}
y &= 3x + 2 \\
y &= -2x + 7 \\
\end{align*}
\]

3.

\[
\begin{align*}
5x - 5y &= 5 \\
5x + 5y &= 35 \\
\end{align*}
\]

4.

\[
\begin{align*}
y &= -3x - 3 \\
3x - 2y + 12 &= 0 \\
\end{align*}
\]

5.

\[
\begin{align*}
3x - 4y &= 3 \\
4y + 5x &= 10 \\
\end{align*}
\]

6.

\[
\begin{align*}
9x - 2y &= -4 \\
2x - 6y &= 1 \\
\end{align*}
\]
7. Supplementary angles are two angles whose sum is $180^\circ$. Angles $A$ and $B$ are supplementary angles. The measure of Angle $A$ is $18^\circ$ less than twice the measure of Angle $B$. Find the measure of each angle.

8. A farmer has fertilizer in 5% and 15% solutions. How much of each type should he mix to obtain 100 liters of fertilizer in a 12% solution?

9. A 150-yard pipe is cut to provide drainage for two fields. If the length of one piece is three yards less than twice the length of the second piece, what are the lengths of the two pieces?

10. Mr. Stein invested a total of $100,000 in two companies for a year. Company A’s stock showed a 13% annual gain, while Company B showed a 3% loss for the year. Mr. Stein made an 8% return on his investment over the year. How much money did he invest in each company?
Here you’ll learn how consistent, inconsistent, and dependent systems arise in real-world applications and you’ll solve such problems.

What if you were playing a game in which you collected houses and hotels. Three houses and one hotel are worth $1750. One house and three hotels are worth $3250. How could you find the value of each house and each hotel? After completing this Concept, you’ll be able to solve real-world applications like this one that involve linear systems.

Watch This

CK-12 Foundation: 0709S Applications of Linear Systems

Guidance

In this section, we’ll see how consistent, inconsistent and dependent systems might arise in applications.

Example A

The movie rental store CineStar offers customers two choices. Customers can pay a yearly membership of $45 and then rent each movie for $2 or they can choose not to pay the membership fee and rent each movie for $3.50. How many movies would you have to rent before the membership becomes the cheaper option?

Solution

Let’s translate this problem into algebra. Since there are two different options to consider, we can write two different equations and form a system.

The choices are “membership” and “no membership.” We’ll call the number of movies you rent \(x\) and the total cost of renting movies for a year \(y\).

<table>
<thead>
<tr>
<th></th>
<th>flat fee</th>
<th>rental fee</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>membership</td>
<td>$45</td>
<td>2(x)</td>
<td>(y = 45 + 2x)</td>
</tr>
<tr>
<td>no membership</td>
<td>$0</td>
<td>3.50(x)</td>
<td>(y = 3.5x)</td>
</tr>
</tbody>
</table>

The flat fee is the dollar amount you pay per year and the rental fee is the dollar amount you pay when you rent a movie. For the membership option the rental fee is 2\(x\), since you would pay $2 for each movie you rented; for the
no membership option the rental fee is 3.50x, since you would pay $3.50 for each movie you rented. 

Our system of equations is:

\[
\begin{align*}
  y &= 45 + 2x \\
  y &= 3.50x
\end{align*}
\]

Here’s a graph of the system:

Now we need to find the exact intersection point. Since each equation is already solved for \( y \), we can easily solve the system with substitution. Substitute the second equation into the first one:

\[
\begin{align*}
  y &= 45 + 2x \\
  3.50x &= 45 + 2x \\
  1.50x &= 45 \\
  x &= 30 \text{ movies}
\end{align*}
\]

You would have to rent **30 movies per year** before the membership becomes the better option.

This example shows a real situation where a consistent system of equations is useful in finding a solution. Remember that for a consistent system, the lines that make up the system intersect at single point. In other words, the lines are not parallel or the slopes are different.

In this case, the slopes of the lines represent the price of a rental per movie. The lines cross because the price of rental per movie is different for the two options in the problem.

Now let’s look at a situation where the system is inconsistent. From the previous explanation, we can conclude that the lines will not intersect if the slopes are the same (and the \( y \)-intercept is different). Let’s change the previous problem so that this is the case.

**Example B**

*Two movie rental stores are in competition. Movie House charges an annual membership of $30 and charges $3 per movie rental. Flicks for Cheap charges an annual membership of $15 and charges $3 per movie rental. After how many movie rentals would Movie House become the better option?*
Solution

It should already be clear to see that Movie House will never become the better option, since its membership is more expensive and it charges the same amount per movie as Flicks for Cheap.

The lines on a graph that describe each option have different \( y \)-intercepts—namely 30 for Movie House and 15 for Flicks for Cheap—but the same slope: 3 dollars per movie. This means that the lines are parallel and so the system is inconsistent.

Now let’s see how this works algebraically. Once again, we’ll call the number of movies you rent \( x \) and the total cost of renting movies for a year \( y \).

<table>
<thead>
<tr>
<th>Table 2.6:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>flat fee</strong></td>
</tr>
<tr>
<td>Movie House</td>
</tr>
<tr>
<td>Flicks for Cheap</td>
</tr>
</tbody>
</table>

The system of equations that describes this problem is:

\[
\begin{align*}
y &= 30 + 3x \\
y &= 15 + 3x
\end{align*}
\]

Let’s solve this system by substituting the second equation into the first equation:

\[
\begin{align*}
y &= 30 + 3x \\
\Rightarrow 15 + 3x &= 30 + 3x & \Rightarrow 15 = 30 & \text{This statement is always false.} \\
y &= 15 + 3x
\end{align*}
\]

This means that the system is inconsistent.

Example C

*Peter buys two apples and three bananas for $4. Nadia buys four apples and six bananas for $8 from the same store. How much does one banana and one apple costs?*

Solution

We must write two equations: one for Peter’s purchase and one for Nadia’s purchase.

Let’s say \( a \) is the cost of one apple and \( b \) is the cost of one banana.

<table>
<thead>
<tr>
<th>Table 2.7:</th>
</tr>
</thead>
<tbody>
<tr>
<td><strong>cost of apples</strong></td>
</tr>
<tr>
<td>Peter</td>
</tr>
<tr>
<td>Nadia</td>
</tr>
</tbody>
</table>

The system of equations that describes this problem is:
2.11. Applications of Linear Systems

\begin{align*}
2a + 3b &= 4 \\
4a + 6b &= 8
\end{align*}

Let’s solve this system by multiplying the first equation by -2 and adding the two equations:

\[-2(2a + 3b = 4) \quad \Rightarrow \quad -4a - 6b = -8\]

\[4a + 6b = 8\]

\[0 + 0 = 0\]

This statement is always true. This means that the system is dependent.

Looking at the problem again, we can see that we were given exactly the same information in both statements. If Peter buys two apples and three bananas for $4, it makes sense that if Nadia buys twice as many apples (four apples) and twice as many bananas (six bananas) she will pay twice the price ($8). Since the second equation doesn’t give us any new information, it doesn’t make it possible to find out the price of each fruit.

Watch this video for help with the Examples above.

CK-12 Foundation: Applications of Linear Systems

Vocabulary

• A linear system of equations is a set of equations that must be solved together to find the one solution that fits them both.

• Solving linear systems by substitution means to solve for one variable in one equation, and then to substitute it into the other equation, solving for the other variable.

• The purpose of the elimination method to solve a system is to cancel, or eliminate, a variable by either adding or subtracting the two equations. Sometimes the equations must be multiplied by scalars first, in order to cancel out a variable.

• A consistent system will always give exactly one solution.

• An inconsistent system will yield a statement that is always false (like 0 = 13).

• A dependent system will yield a statement that is always true (like 9 = 9).
Guided Practice

A baker sells plain cakes for $7 and decorated cakes for $11. On a busy Saturday the baker started with 120 cakes, and sold all but three. His takings for the day were $991. How many plain cakes did he sell that day, and how many were decorated before they were sold?

Solution:

<table>
<thead>
<tr>
<th>Cakes sold</th>
<th>plain cakes</th>
<th>decorated cakes</th>
<th>total</th>
</tr>
</thead>
<tbody>
<tr>
<td>Cost of cakes</td>
<td>p</td>
<td>d</td>
<td>120 − 3 = 117</td>
</tr>
<tr>
<td></td>
<td>7p</td>
<td>11d</td>
<td>$991</td>
</tr>
</tbody>
</table>

The system of equations that describes this problem is:

\[ p + d = 117 \]
\[ 7p + 11d = 991 \]

Let’s solve this system by substituting the second equation into the first equation:

\[ p + d = 117 \Rightarrow p = 117 − d \]

\[ 7p + 11d = 991 \Rightarrow 7(117 − d) + 11d = 991 \]
\[ \Rightarrow 819 − 7d + 11d = 991 \]
\[ \Rightarrow 819 + 4d = 991 \]
\[ \Rightarrow 4d = 172 \]
\[ \Rightarrow d = 43 \]

We can substitute \( d \) into the first equation to solve for \( p \).

\[ p = 117 − d = 117 − (43) = 74 \]

The baker sold 74 plain cakes and 43 decorated cakes.

Practice

1. Twice John’s age plus five times Claire’s age is 204. Nine times John’s age minus three times Claire’s age is also 204. How old are John and Claire?
2. Juan is considering two cell phone plans. The first company charges $120 for the phone and $30 per month for the calling plan that Juan wants. The second company charges $40 for the same phone but charges $45 per month for the calling plan that Juan wants. After how many months would the total cost of the two plans be the same?
3. Jamal placed two orders with an internet clothing store. The first order was for 13 ties and 4 pairs of suspenders, and totaled $487. The second order was for 6 ties and 2 pairs of suspenders, and totaled $232. The bill does not list the per-item price, but all ties have the same price and all suspenders have the same price. What is the cost of one tie and of one pair of suspenders?
4. An airplane took four hours to fly 2400 miles in the direction of the jet-stream. The return trip against the jet-stream took five hours. What were the airplane’s speed in still air and the jet-stream’s speed?

For questions 5-7, a movie theater charges $4.50 for children and $8.00 for adults.

5. On a certain day, 1200 people enter the theater and $8375 is collected. How many children and how many adults attended?
6. The next day, the manager announces that she wants to see them take in $10000 in tickets. If there are 240 seats in the house and only five movie showings planned that day, is it possible to meet that goal?
7. At the same theater, a 16-ounce soda costs $3 and a 32-ounce soda costs $5. If the theater sells 12,480 ounces of soda for $2100, how many people bought soda? (Note: Be careful in setting up this problem!)

For questions 8-10, consider the situation: Nadia told Peter that she went to the farmer’s market and bought two apples and one banana, and that it cost her $2.50. She thought that Peter might like some fruit, so she went back to the seller and bought four more apples and two more bananas. Peter thanked Nadia, but told her that he did not like bananas, so he would only pay her for four apples. Nadia told him that the second time she paid $6.00 for the fruit.

8. What did Peter find when he tried to figure out the price of four apples?
9. Nadia then told Peter she had made a mistake, and she actually paid $5.00 on her second trip. Now what answer did Peter get when he tried to figure out how much to pay her?
10. Alicia then showed up and told them she had just bought 3 apples and 2 bananas from the same seller for $4.25. Now how much should Peter pay Nadia for four apples?
2.12 Linear Programming

Here you’ll learn how to analyze and find the feasible solution(s) to a system of inequalities under a given set of constraints.

What if you had an equation like \( z = x + y \) in which a set of constraints like \( x - y \leq 4 \), \( x + y \leq 2 \), and \( 2x + 3y \geq -3 \) were placed on it. How could you find the minimum and maximum values of \( z \)? After completing this Concept, you’ll be able to analyze a system of inequalities to make the best decisions given the constraints of the situation.

Watch This

CK-12 Foundation: 0711S Linear Programming

Graphing calculators can be very useful for problems that involve this many inequalities. The following video shows a real-world linear programming problem worked through in detail on a graphing calculator, although the methods used there can also be used for pencil-and-paper solving.

Stacy Reagan: Linear Programming

Guidance

A lot of interesting real-world problems can be solved with systems of linear inequalities.

For example, you go to your favorite restaurant and you want to be served by your best friend who happens to work there. However, your friend only waits tables in a certain region of the restaurant. The restaurant is also known for its great views, so you want to sit in a certain area of the restaurant that offers a good view. Solving a system of linear inequalities will allow you to find the area in the restaurant where you can sit to get the best view and be served by your friend.

Often, systems of linear inequalities deal with problems where you are trying to find the best possible situation given a set of constraints. Most of these application problems fall in a category called linear programming problems.

Linear programming is the process of taking various linear inequalities relating to some situation, and finding the best possible value under those conditions. A typical example would be taking the limitations of materials and labor
at a factory, then determining the best production levels for maximal profits under those conditions. These kinds of problems are used every day in the organization and allocation of resources. These real-life systems can have dozens or hundreds of variables, or more. In this section, we’ll only work with the simple two-variable linear case.

The general process is to:

- Graph the inequalities (called constraints) to form a bounded area on the coordinate plane (called the feasibility region).
- Figure out the coordinates of the corners (or vertices) of this feasibility region by solving the system of equations that applies to each of the intersection points.
- Test these corner points in the formula (called the optimization equation) for which you’re trying to find the maximum or minimum value.

**Example A**

*If $z = 2x + 5y$, find the maximum and minimum values of $z$ given these constraints:*

$$
2x - y \leq 12 \\
4x + 3y \geq 0 \\
x - y \geq 6
$$

**Solution**

First, we need to find the solution to this system of linear inequalities by graphing and shading appropriately. To graph the inequalities, we rewrite them in slope-intercept form:

$$
y \geq 2x - 12 \\
y \geq -\frac{4}{3}x \\
y \leq x - 6
$$

These three linear inequalities are called the constraints, and here is their graph:
The shaded region in the graph is called the **feasibility region**. All possible solutions to the system occur in that region; now we must try to find the maximum and minimum values of the variable \( z \) within that region. In other words, which values of \( x \) and \( y \) within the feasibility region will give us the greatest and smallest overall values for the expression \( 2x + 5y \)?

Fortunately, we don’t have to test every point in the region to find that out. It just so happens that the minimum or maximum value of the optimization equation in a linear system like this will always be found at one of the vertices (the corners) of the feasibility region; we just have to figure out which vertices. So for each vertex—each point where two of the lines on the graph cross—we need to solve the system of just those two equations, and then find the value of \( z \) at that point.

**The first system** consists of the equations \( y = 2x - 12 \) and \( y = -\frac{4}{3}x \). We can solve this system by substitution:

\[
-\frac{4}{3}x = 2x - 12 \Rightarrow -4x = 6x - 36 \Rightarrow -10x = -36 \Rightarrow x = 3.6
\]

\[
y = 2x - 12 \Rightarrow y = 2(3.6) - 12 \Rightarrow y = -4.8
\]

The lines intersect at the point (3.6, -4.8).

**The second system** consists of the equations \( y = 2x - 12 \) and \( y = x - 6 \). Solving this system by substitution:

\[
x - 6 = 2x - 12 \Rightarrow 6 = x \Rightarrow x = 6
\]

\[
y = x - 6 \Rightarrow y = 6 - 6 \Rightarrow y = 0
\]

The lines intersect at the point (6, 0).

**The third system** consists of the equations \( y = -\frac{4}{3}x \) and \( y = x - 6 \). Solving this system by substitution:

\[
x - 6 = -\frac{4}{3}x \Rightarrow 3x - 18 = -4x \Rightarrow 7x = 18 \Rightarrow x = 2.57
\]

\[
y = x - 6 \Rightarrow y = 2.57 - 6 \Rightarrow y = -3.43
\]
The lines intersect at the point (2.57, -3.43).

So now we have three different points that might give us the maximum and minimum values for \(z\). To find out which ones actually do give the maximum and minimum values, we can plug the points into the optimization equation \(z = 2x + 5y\).

When we plug in (3.6, -4.8), we get \(z = 2(3.6) + 5(-4.8) = -16.8\).

When we plug in (6, 0), we get \(z = 2(6) + 5(0) = 12\).

When we plug in (2.57, -3.43), we get \(z = 2(2.57) + 5(-3.43) = -12.01\).

So we can see that the point (6, 0) gives us the maximum possible value for \(z\) and the point (3.6, -4.8) gives us the minimum value.

In the previous example, we learned how to apply the method of linear programming in the abstract. In the next example, we’ll look at a real-life application.

**Example B**

You have $10,000 to invest, and three different funds to choose from. The municipal bond fund has a 5% return, the local bank’s CDs have a 7% return, and a high-risk account has an expected 10% return. To minimize risk, you decide not to invest any more than $1,000 in the high-risk account. For tax reasons, you need to invest at least three times as much in the municipal bonds as in the bank CDs. What’s the best way to distribute your money given these constraints?

**Solution:**

Let’s define our variables:

- \(x\) is the amount of money invested in the municipal bond at 5% return
- \(y\) is the amount of money invested in the bank’s CD at 7% return
- \(10000 - x - y\) is the amount of money invested in the high-risk account at 10% return
- \(z\) is the total interest returned from all the investments, so \(z = .05x + .07y + .1(10000 - x - y)\) or \(z = 1000 - 0.05x - 0.03y\). This is the amount that we are trying to maximize. Our goal is to find the values of \(x\) and \(y\) that maximizes the value of \(z\).

Now, let’s write inequalities for the *constraints*:

You decide not to invest more than $1000 in the high-risk account—that means:

\[
10000 - x - y \leq 1000
\]

You need to invest at least three times as much in the municipal bonds as in the bank CDs—that means:

\[
3y \leq x
\]

Also, you can’t invest less than zero dollars in each account, so:

\[
x \geq 0
\]

\[
y \geq 0
\]

\[
10000 - x - y \geq 0
\]
To summarize, we must maximize the expression \( z = 1000 - .05x - .03y \) using the constraints:

\[
egin{align*}
10000 - x - y &\leq 1000 \\
3y &\leq x \\
x &\geq 0 \\
y &\geq 0 \\
10000 - x - y &\geq 0 \\
y &\geq 9000 - x \\
y &\leq \frac{x}{3} \\
x &\geq 0 \\
y &\geq 0 \\
y &\leq 10000 - x
\end{align*}
\]

**Step 1:** Find the solution region to the set of inequalities by graphing each line and shading appropriately. The following figure shows the overlapping region:

![Graph showing the overlapping region](image)

The purple region is the feasibility region where all the possible solutions can occur.

**Step 2:** Next we need to find the corner points of the feasibility region. Notice that there are four corners. To find their coordinates, we must pair up the relevant equations and solve each resulting system.

**System 1:**

\[
\begin{align*}
y &= \frac{x}{3} \\
y &= 10000 - x
\end{align*}
\]

Substitute the first equation into the second equation:

\[
\begin{align*}
\frac{x}{3} &= 10000 - x \\
\Rightarrow x &= 30000 - 3x \\
\Rightarrow 4x &= 30000 \\
\Rightarrow x &= 7500
\end{align*}
\]

\[
\begin{align*}
y &= \frac{x}{3} \\
\Rightarrow y &= \frac{7500}{3} \\
\Rightarrow y &= 2500
\end{align*}
\]

The intersection point is \((7500, 2500)\).

**System 2:**

(Proceed with system 2 calculations if needed.)
\[ y = \frac{x}{3} \]

\[ y = 9000 - x \]

Substitute the first equation into the second equation:

\[ \frac{x}{3} = 9000 - x \Rightarrow x = 27000 - 3x \Rightarrow 4x = 27000 \Rightarrow x = 6750 \]

\[ y = \frac{x}{3} \Rightarrow y = \frac{6750}{3} \Rightarrow y = 2250 \]

The intersection point is (6750, 2250).

*System 3:*

\[ y = 0 \]

\[ y = 10000 - x \]

The intersection point is (10000, 0).

*System 4:*

\[ y = 0 \]

\[ y = 9000 - x \]

The intersection point is (9000, 0).

**Step 3:** In order to find the maximum value for \( z \), we need to plug all the intersection points into the equation for \( z \) and find which one yields the largest number.

(7500, 2500): \( z = 1000 - 0.05(7500) - 0.03(2500) = 550 \)

(6750, 2250): \( z = 1000 - 0.05(6750) - 0.03(2250) = 595 \)

(10000, 0): \( z = 1000 - 0.05(10000) - 0.03(0) = 500 \)

(9000, 0): \( z = 1000 - 0.05(9000) - 0.03(0) = 550 \)

The maximum return on the investment of $595 occurs at the point (6750, 2250). This means that:

$6,750 is invested in the municipal bonds.

$2,250 is invested in the bank CDs.

$1,000 is invested in the high-risk account.

**Example C**

James is trying to expand his pastry business to include cupcakes and personal cakes. He has 40 hours available to decorate the new items and can use no more than 22 pounds of cake mix. Each personal cake requires 2 pounds of cake mix and 2 hours to decorate. Each cupcake order requires one pound of cake mix and 4 hours to decorate. If he can sell each personal cake for $14.99 and each cupcake order for $16.99, how many personal cakes and cupcake orders should James make to make the most revenue?
There are four inequalities in this situation. First, state the variables. Let \( p = \text{the number of personal cakes} \) and \( c = \text{the number of cupcake orders} \).

Translate this into a system of inequalities.

\[
2p + 1c \leq 22 \quad \text{– This is the amount of available cake mix.}
\]

\[
2p + 4c \leq 40 \quad \text{– This is the available time to decorate.}
\]

\[
p \geq 0 \quad \text{– You cannot make negative personal cakes.}
\]

\[
c \geq 0 \quad \text{– You cannot make negative cupcake orders.}
\]

Now graph each inequality and determine the feasible region.

The feasible region has four vertices: \{(0, 0),(0, 10),(11, 0),(8, 6)\}. According to our theorem, the optimization answer will only occur at one of these vertices.

Write the optimization equation. How much of each type of order should James make to bring in the most revenue?

\[
14.99p + 16.99c = \text{maximum revenue}
\]

Substitute each ordered pair to determine which makes the most money.

\[
\begin{align*}
(0, 0) & \rightarrow 0.00 \\
(0, 10) & \rightarrow 14.99(0) + 16.99(10) = 169.90 \\
(11, 0) & \rightarrow 14.99(11) + 16.99(0) = 164.89 \\
(8, 6) & \rightarrow 14.99(8) + 16.99(6) = 221.86
\end{align*}
\]

To make the most revenue, James should make 8 personal cakes and 6 cupcake orders.

Watch this video for help with the Examples above.
CK-12 Foundation: Linear Programming

Vocabulary

- **Linear programming** is the mathematical process of analyzing a system of inequalities to make the best decisions given the constraints of the situation.

- **Constraints** are the particular restrictions of a situation due to time, money, or materials.

- In an **optimization** problem, the goal is to locate the feasible region of the system and use it to answer a profitability, or **optimization**, question.

- The **solution for the system of inequalities** is the common shaded region between all the inequalities in the system.

- The common shaded region of the system of inequalities is called the **feasible region**.

Guided Practice

*Graph the solution to the following system:*

\[
\begin{align*}
x - y &< -6 \\
2y &\geq 3x + 17
\end{align*}
\]

**Solution:**
First we will rewrite the equations in slope-intercept form in order to graph them:

**Inequality 1**

\[
x - y < -6 \quad \text{Solve for } y.
\]

\[
-y < -x - 6 \quad \text{Subtract } x \text{ from each side.}
\]

\[
y > x + 6 \quad \text{Multiply each side by -1, flipping the inequality.}
\]

**Inequality 2**

\[
2y \geq 3x + 17 < \quad \text{Solve for } y.
\]

\[
y \geq \frac{3}{2}x + 8.5 < \quad \text{Divide each side by } 2.
\]

Graph each equation and shade accordingly:
Practice

Solve the following linear programming problems.

1. Given the following constraints, find the maximum and minimum values of \( z = -x + 5y \):

\[
\begin{align*}
  x + 3y & \leq 0 \\
  x - y & \geq 0 \\
  3x - 7y & \leq 16
\end{align*}
\]

Santa Claus is assigning elves to work an eight-hour shift making toy trucks. Apprentice elves draw a wage of five candy canes per hour worked, but can only make four trucks an hour. Senior elves can make six trucks an hour and are paid eight candy canes per hour. There’s only room for nine elves in the truck shop, and due to a candy-makers’ strike, Santa Claus can only pay out 480 candy canes for the whole 8-hour shift.

2. How many senior elves and how many apprentice elves should work this shift to maximize the number of trucks that get made?
3. How many trucks will be made?
4. Just before the shift begins, the apprentice elves demand a wage increase; they insist on being paid seven candy canes an hour. Now how many apprentice elves and how many senior elves should Santa assign to this shift?
5. How many trucks will now get made, and how many candy canes will Santa have left over?

In Adrian’s Furniture Shop, Adrian assembles both bookcases and TV cabinets. Each type of furniture takes her about the same time to assemble. She figures she has time to make at most 18 pieces of furniture by this Saturday. The materials for each bookcase cost her $20 and the materials for each TV stand costs her $45. She has $600 to spend on materials. Adrian makes a profit of $60 on each bookcase and a profit of $100 on each TV stand.

6. Set up a system of inequalities. What \( x \)− and \( y \)−values do you get for the point where Adrian’s profit is maximized? Does this solution make sense in the real world?
7. What two possible real-world \( x \)−values and what two possible real-world \( y \)−values would be closest to the values in that solution?
8. With two choices each for \( x \) and \( y \), there are four possible combinations of \( x \)− and \( y \)−values. Of those four combinations, which ones actually fall within the feasibility region of the problem?
9. Which one of those feasible combinations seems like it would generate the most profit? Test out each one to confirm your guess. How much profit will Adrian make with that combination?
10. Based on Adrian’s previous sales figures, she doesn’t think she can sell more than 8 TV stands. Now how many of each piece of furniture should she make, and what will her profit be?

11. Suppose Adrian is confident she can sell all the furniture she can make, but she doesn’t have room to display more than 7 bookcases in her shop. Now how many of each piece of furniture should she make, and what will her profit be?

12. Here’s a “linear programming” problem on a line instead of a plane: Given the constraints $x \leq 5$ and $x \geq -2$, maximize the value of $y$ where $y = x + 3$.

**Texas Instruments Resources**

*In the CK-12 Texas Instruments Algebra I FlexBook, there are graphing calculator activities designed to supplement the objectives for some of the lessons in this chapter. See [http://www.ck12.org/flexr/chapter/9617](http://www.ck12.org/flexr/chapter/9617).*
2.13 Graphs of Absolute Value Equations

Here you’ll learn how to make a table of values for absolute value functions so you can graph them. What if you were given an absolute value function like \( y = |x - 8| \)? How could you graph this function? After completing this Concept, you’ll be able to make a table of values to graph absolute value functions like this one.

Watch This

Guidance

Now let’s look at how to graph absolute value functions.

Example A

Consider the function \( y = |x - 1| \). We can graph this function by making a table of values:

| \( x \) | \( y = |x - 1| \) |
|---|---|
| -2 | \( y = |-2 - 1| = |-3| = 3 \) |
| -1 | \( y = |-1 - 1| = |-2| = 2 \) |
| 0  | \( y = |0 - 1| = |-1| = 1 \) |
| 1  | \( y = |1 - 1| = |0| = 0 \) |
| 2  | \( y = |2 - 1| = |1| = 1 \) |
| 3  | \( y = |3 - 1| = |2| = 2 \) |
| 4  | \( y = |4 - 1| = |3| = 3 \) |
2.13. Graphs of Absolute Value Equations

You can see that the graph of an absolute value function makes a big “V”. It consists of two line rays (or line segments), one with positive slope and one with negative slope, joined at the vertex or cusp.

We’ve already seen that to solve an absolute value equation we need to consider two options:

1. The expression inside the absolute value is not negative.
2. The expression inside the absolute value is negative.

Combining these two options gives us the two parts of the graph.

For instance, in the above example, the expression inside the absolute value sign is $x - 1$. By definition, this expression is nonnegative when $x - 1 \geq 0$, which is to say when $x \geq 1$. When the expression inside the absolute value sign is nonnegative, we can just drop the absolute value sign. So for all values of $x$ greater than or equal to 1, the equation is just $y = x - 1$.

On the other hand, when $x - 1 < 0$ —in other words, when $x < 1$ —the expression inside the absolute value sign is negative. Since absolute value respresents distance which is always positive, we need the opposite of what is inside. That means we have to drop the absolute value sign but also multiply the expression by -1. So for all values of $x$ less than 1, the equation is $y = -(x - 1)$, or $y = -x + 1$.

These are both graphs of straight lines, as shown above. They meet at the point where $x - 1 = 0$ —that is, at $x = 1$.

We can graph absolute value functions by breaking them down algebraically as we just did, or we can graph them using a table of values. However, when the absolute value equation is linear, the easiest way to graph it is to combine those two techniques, as follows:

1. Find the vertex of the graph by setting the expression inside the absolute value equal to zero and solving for $x$.
2. Make a table of values that includes the vertex, a value smaller than the vertex, and a value larger than the vertex. Calculate the corresponding values of $y$ using the equation of the function.
3. Plot the points and connect them with two straight lines that meet at the vertex.

**Example B**

*Graph the absolute value function $y = |x + 5|$.*

**Solution**

*Step 1:* Find the vertex by solving $x + 5 = 0$. The vertex is at $x = -5$.

*Step 2:* Make a table of values:
Step 3: Plot the points and draw two straight lines that meet at the vertex.

Example C

**Graph the absolute value function:** \( y = |3x - 12| \)

**Solution**

*Step 1:* Find the vertex by solving \( 3x - 12 = 0 \). The vertex is at \( x = 4 \).

*Step 2:* Make a table of values:

| \( x \)  | \( y = |3x - 12| \)   |
|-------|---------------------|
| 0     | \( y = |3(0) - 12| = 12 \) |
| 4     | \( y = |3(4) - 12| = 0 \) |
| 8     | \( y = |3(8) - 12| = 12 \) |

Step 3: Plot the points and draw two straight lines that meet at the vertex.
2.13. Graphs of Absolute Value Equations

Watch this video for help with the Examples above.

CK-12 Foundation: Graphs of Absolute Value Equations

Vocabulary

- The absolute value of a number is its distance from zero on a number line.
- \(|x| = x\) if \(x\) is not negative, and \(|x| = -x\) if \(x\) is negative.
- An equation or inequality with an absolute value in it splits into two equations, one where the expression inside the absolute value sign is positive and one where it is negative. When the expression within the absolute value is positive, then the absolute value signs do nothing and can be omitted. When the expression within the absolute value is negative, then the expression within the absolute value signs must be negated before removing the signs.
- Inequalities of the type \(|x| < a\) can be rewritten as “\(-a < x < a\)”.
- Inequalities of the type \(|x| > b\) can be rewritten as “\(x < -b\) or \(x > b\)”.

Guided Practice

*Graph the absolute value function: \(y = 3|x - 4|\)*

*Solution*

*Step 1:* Find the vertex by solving \(x - 4 = 0\). The vertex is at \(x = 4\).

*Step 2:* Make a table of values:

| \(x\) | \(y = 3|x - 4|\) |
|-------|------------------|
| 0     | \(y = 3|0 - 4| = 3 \cdot 4 = 12\) |
| 4     | \(y = 3|4 - 4| = 3|0| = 3 \cdot 0 = 0\) |
Notice this is the same table as Example C. The function \( y = 3|x - 4| \) is equivalent to the function \( y = |3x - 12| \). This is because positive numbers can be factored out, or distributed into the absolute value function.

**Step 3:** Plot the points and draw two straight lines that meet at the vertex.

![Graph of absolute value function](image)

**Practice**

Graph the absolute value functions.

1. \( y = |x + 3| \)
2. \( y = |x - 6| \)
3. \( y = |4x + 2| \)
4. \( y = |5 - 6x| \)
5. \( y = |2x - 1| \)
6. \( y = |3|2x - 7| \)
7. \( y = 0.05|x - 1.25| \)
8. \( y = \frac{1}{5}|x + 10| \)
9. \( y = |\frac{x}{3} - 4| \)
10. \( y = -2\left|\frac{x}{2} - 5\right| \)
### 2.14 Body Temperature (Absolute Value Inequalities)

#### Real World Applications – Algebra I

**Topic**

What’s normal for a person’s body temperature?

**Student Exploration**

It’s common knowledge that a person’s normal body temperature is supposed to be 98.6 degrees. We can figure out using absolute value equations how much a person’s body temperature deviates from the norm for it to be considered abnormal (and possibly sick). Physicians say that people’s body temperature shouldn’t exceed 0.5 degrees from the norm. How can we represent this relationship as an absolute value equation, and then solve to know what the minimum and maximum body temperatures are?

Let’s say that “t” represents a person’s normal body temperature.

\[ |t - 98.6| = 0.5 \]

This inequality means that the normal body temperature subtracted from the minimum and maximum body temperature should equal 0.5.

To solve this, we can break this up into two equations.

\[ t - 98.6 = 0.5 \text{ and } t - 98.6 = -0.5 \]

\[ t = 98.6 + 0.5 \text{ and } t = 98.6 - 0.5 \]

\[ t = 99.1 \text{ and } t = 98.1 \]

This means that our normal body temperature should be between 98.1 and 99.1 degrees.

We can also graph this absolute value equation and see visually see what it means. Since we solved this equation, we can graph our solution set on a number line. We would also represent our solution space between the 98.1 and 99.1 tick marks on the number line. The solution space represents all of the different temperatures that are “normal” for humans.

We can also graph the solution space on an xy coordinate graph, and interpret the solution. For this relationship, we’d have to graph two separate equations: \( y = |x - 98.6| \) and \( y = 0.5 \) See below.
The horizontal line represents the variant of the normal body temperature. The intersection between the “V” graph and the horizontal line is our solution 98. and 99.1.

A few steps further: What does the point of the “V” graph represent on the graph? What do the x values represent, in relation to body temperature? What do the y values represent? If we were to shade the inside of the “V” below the horizontal line, what would the solution space represent?

Let’s explore a little bit more deeply into this body temperature relationship and integrate absolute value inequalities in the equations and graphs. If we first had to integrate the use of an inequality sign instead of the equation $|t - 98.6| = 0.5$, should we use a greater than sign, or a less than sign? Why?

Our inequality would be $|t - 98.6| \leq 0.5$ because the body temperature difference can’t be higher than 0.5 variance. If we were to represent this on a number line, we would have our solution space in between the endpoints at 98.1 and 99.1. This represents all of the temperatures that are considered “normal.”

**Extension Investigation**

How else can you represent the maximum and minimum of something as an absolute value equation?